UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics C315: Numerical Analysis I

COURSE CODE : **MATHC315**

UNIT VALUE : **0.50**

DATE : **11-MAY-04**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

04-C0916-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the following numerical scheme with step-size h for solving the initial value problem consisting of the ordinary differential equation $dy/dt = f(y,t)$ and the initial condition $y(0) = Y_0$,

$$y_{n+1} = y_n + \alpha h f(y_n, t_n) + (1-\alpha) h f(y_{n+1}, t_{n+1}), \qquad y_0 = Y_0, \qquad (*)$$

where $0 \leq \alpha \leq 1$.

(a) Explain the distinction between implicit and explicit numerical schemes. For which values of α is the scheme above
(i) implicit
(ii) explicit.

(b) Define what is meant by the concept of stability in the case of a generic numerical scheme $y_{n+1} = \tau(y_n, t_n)$.

(c) By considering the linear equation $dy/dt = \lambda y$, $\lambda \in C$, derive and sketch the regions of absolute stability for the method $(*)$ in the complex λh plane when (i) $\alpha = 1$, (ii) $\alpha = 0$. Define what is meant by an A-stable method and state whether the method is A-stable in each case.

(d) Show that the scheme $(*)$ is A-stable when $\alpha = 1/2$.

2. The function

$$S(x) = \begin{cases} a_0 + b_0 x + c_0 x^2 + d_0 x^3, & x \in [0,1], \\ a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3, & x \in [1,2], \\ a_2 + b_2(x-2) + c_2(x-2)^2 + d_2(x-2)^3, & x \in [2,3], \end{cases}$$

interpolates the set of points $\{(0,0),(1,3),(2,9),(3,12)\}$.

(a) Write the twelve conditions that the twelve unknown constants must satisfy in order for $S(x)$ to be a natural cubic spline function (i.e. with $S''(0) = S''(3) = 0$) on the interval $x \in [0,3]$.

(b) Eliminate the a_j, b_j, and d_j in favour of three equations in the c_j.

(c) By writing the remaining three equations in matrix form and using Gaussian elimination or otherwise, determine the value of each of the constants.

3. The $N+1$ times continuously differentiable function $y(x)$ is to be interpolated at a set of $N+1$ distinct points $(x_i, y(x_i))$, where $i = 0, 1, ..., N$.

(a) Show that there exists a unique Nth order polynomial $P_N(x)$ that interpolates this set of points.

(b) By considering the properties of the function

$$\phi(t) = y(t) - P_N(t) - [y(\overline{x}) - P_N(\overline{x})] \prod_{k=0}^{N} \frac{t - x_j}{\overline{x} - x_j}, \qquad \overline{x}, t \in [x_0, x_N],$$

or otherwise, show that there exists $\zeta \in [x_0, x_N]$ such that

$$y(\overline{x}) - P_N(\overline{x}) = \frac{y^{(N+1)}(\zeta)}{(N+1)!} \prod_{j=0}^{N} (\overline{x} - x_j).$$

(c) Hence find an upper bound for the error when a quadratic polynomial is used to interpolate $y(x) = \cosh x$ on the interval [0,1] using an evenly spaced grid.

4. (a) Show that the Newton-Raphson iteration for finding a root of the continuously differentiable function $f(x)$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

may be derived from consideration of the equation of the tangent to the curve $y = f(x)$ at $(x_n, f(x_n))$.

(b) By considering the error at the nth step $e_n = x_n - r$, show that the Newton-Raphson iteration converges quadratically to a simple root at $x = r$.

(c) Consider the function

$$f(x) = x^\alpha, \qquad \alpha > 1.$$

Show that for this example, the Newton-Raphson iteration takes the form of a linear first order difference equation. Write down the general solution of this equation.

(d) From a starting point $x_0 = 1$, estimate how many steps are necessary for the iteration to converge to the root at $x = 0$ within an accuracy $\delta = e^{-10}$ when $\alpha = 1000$. Comment on the order of convergence of the method in this example. Why are so many iterations necessary for this function?

5. Consider the initial value problem

$$\frac{dy}{dt} = f(y,t), \qquad y(0) = y_0.$$

(a) Show, by considering the Taylor expansion of $y(t)$, that the second order Runge-Kutta (RK2) method for solving the IVP, with step-size h, may be written

$$y_{n+1} = y_n + hf(y_n, t_n) + \frac{h}{2\lambda}\left[f(y_n + \lambda h f(y_n, t_n), t_n + \lambda h) - f(y_n, t_n) \right],$$

where λ is an arbitrary constant.

(b) Define local truncation error. Derive an expression for the $O(h^3)$ local truncation error of the RK2 method.

(c) Using this expression find the value of λ that optimises the method for

(i) $f(y,t) = F(t)$,

(ii) $f(y,t) = y^\alpha, \qquad (\alpha \neq 1)$.