UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C315: Numerical Analysis I

COURSE CODE	:	**MATHC315**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0909-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Consider the initial value problem for a first order ordinary differential equation

$$\frac{dy}{dt} = f(y), \quad y(t_0) = y_0.$$

(a) Briefly outline the general principles used to derive a Runge-Kutta method involving q evaluations of the function f (no detailed algebra is required).

(b) Show that the scheme

$$\begin{aligned} y_{n+1} &= y_n + \tfrac{1}{4}(w_1 + 2w_2 + w_3) \\ w_1 &= hf(y_n) \\ w_2 &= hf(y_n + \tfrac{2}{3}w_1) \\ w_3 &= hf(y_n - \tfrac{1}{3}w_1 + w_2) \end{aligned}$$

where h is the step-size, is a Runge-Kutta scheme of third order.

(c) Define the growth factor g for a general numerical scheme $y_{n+1} = \tau(y_n, t_n)$. Explain why the stability of the numerical scheme is related to g and show that

$$g \approx \frac{\partial \tau}{\partial y_n}.$$

(d) Examine the stability of the Runge-Kutta scheme above for the case $f(y) = -\lambda y$, (where $\lambda > 0$). Show that the scheme is stable only for $\lambda h < r$, where r is the real root of

$$s^3 - 3s^2 + 6s - 12 = 0.$$

2. The secant method used to generate an iteration to find a root r of the nonlinear equation $f(x) = 0$ may be written

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1. \tag{1}$$

(a) Is there a disadvantage to using this formulation of the secant method when implementing it numerically?

(b) Writing $e_n = x_n - r$ and using Taylor series to expand terms in (1) to the required order, find recurrence relations between e_{n+1}, e_n and e_{n-1} at leading order in the following cases

(i) $f'(r) \neq 0$.
(ii) $f'(r) = 0$, $f''(r) \neq 0$.

(c) Considering the equation from case (i) and assuming that at large n

$$\frac{e_n}{(e_{n-1})^p} \to \frac{e_{n+1}}{(e_n)^p} \to \Gamma \qquad \text{(constant)},$$

solve for p and Γ and state the order of convergence of the method.

(d) What are p and Γ for case (ii)? How would you constrain your initial guesses x_0 and x_1 in this case to help ensure convergence? (It may help to consider the method graphically.)

3. Let r be a root of $f(x) = 0$ in the interval $[a_0, b_0]$.

(a) Explain geometrically the algorithm used by the regula falsi (false position) method for finding r. Hence obtain the formula

$$w = \frac{f(b_n)a_n - f(a_n)b_n}{b_n - a_n}, \qquad n \geq 0,$$

and describe how a_n, b_n are updated using w after each step of the iteration.

(b) Prove that the regula falsi method is linearly convergent, i.e. $e_{n+1}/e_n \to \kappa$ (constant), where e_n is the error in the estimate of the root at the nth step. (You may assume that one of the endpoints remains fixed throughout the iteration.)

(c) Obtain an estimate for the number of iterations needed to obtain r correct to six decimal places when

$$f(x) = x^3 - 2x^2 - x, \qquad x \in [2, 3].$$

(assume $e_0 = 1$ and use $\log_{10} 2 \sim 0.3$, $\log_{10} 3 \sim 0.5$.)

4. Consider the initial value problem for a first order ordinary differential equation

$$\frac{dy}{dt} = f(y,t), \quad y(0) = y_0.$$

The second order Adams-Bashford (AB2) method to solve this problem numerically with step-size h is given by

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(y_n,t_n) - f(y_{n-1},t_{n-1})\right] \qquad n \geq 1.$$

(a) How would you classify this method (e.g. single-step or multi-step, implicit or explicit)? What are the advantages and disadvantages of AB2 compared with a comparable Runge-Kutta method?

(b) Consider the example with $f(y,t) = -y$ and $y_0 = 1$. Show that the AB2 algorithm can be written as a difference equation, and by solving this equation, show that the numerical solution is

$$y_n = A\mu_1^n + B\mu_2^n,$$

where $\mu_1 > 0$ and $\mu_2 < 0$ should be found. Show also that $A + B = 1$.

(c) The second term in the expression above is clearly spurious as it oscillates in sign with every step. Explain briefly how AB2 can be initialised to eliminate this term.

(d) By writing

$$y_n = \mu_1^n = e^{-\lambda t_n}$$

and carefully expanding λ in powers of h, show explicitly that to leading order in h the global truncation error is

$$e_n = y_n - y(t_n) = \frac{5}{12}h^2 e^{-t_n}.$$

Did you expect the error to be of this order? Why?

$$\left[\text{Reminder:} \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\ldots, \qquad |x| < 1.\right]$$

5. Consider a set of $N+1$ points $\{x_i, y_i\}$ defined on a regular grid $x_i = x_0 + ih$, where $i = 0, ..., N$.

(a) Define a spline function of order m on $[x_0, x_N]$.

(b) Show explicitly that to derive a natural cubic spline, interpolating the given set of points, from the set of polynomial functions

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \qquad x \in [x_j, x_{j+1}],$$

where $j = 0, ..., N-1$, leads to $4N$ equations determining a_j, b_j, c_j and d_j.

(c) Show that these equations may be reduced to the system

$$\mathcal{A}\mathbf{c} = \mathbf{g}$$

where $\mathcal{A}$ is an $N+1 \ \times \ N+1$ tridiagonal matrix, $\mathbf{c}$ is an vector of the quadratic coefficients $\{c_i\}$ and the components of the vector $\mathbf{g}$ are known in terms of $\{y_i\}$.

(d) Outline an algorithm for efficiently solving a general tridiagonal system that could be used to obtain $\mathbf{c}$. How do we know that the matrix $\mathcal{A}$ is not singular?