UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C315: Numerical Analysis I

COURSE CODE	:	**MATHC315**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0907-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) In the context of solving non-linear equations define what is meant by p^{th}-order convergence.

   (b) State the Newton-Raphson method to find a root of $f(x) = 0$ and, by defining $e_n = x_n - r$, show that it converges quadratically to a simple root r.

   (c) For the case of a double root ($f(r) = f'(r) = 0, \quad f''(r) \neq 0$) find the constant β such that the modified Newton-Raphson method

   $$x_{n+1} = x_n - \beta \frac{f(x_n)}{f'(x_n)},$$

   has quadratic convergence to a double root and show that the original Newton-Raphson method converges only linearly in this case.

   (d) Briefly discuss the advantages and disadvantages of the Newton-Raphson method compared to interval refinement methods.

2. (a) Given a set of $N+1$ distinct points (x_i, y_i), $i = 0, 1, \ldots, N$, define what is meant by a spline function of order m on $[x_0, x_N]$.

   (b) Let $s(x)$ be a cubic spline function on $[x_0, x_n]$, such that it interpolates the nodes (x_i, y_i), $i = 0, 1, \ldots, N$. Show, by counting the number of constraints and unknowns in $s(x)$ as defined above, that two additional constraints are required such as the behaviour of the curvature $s''(x)$ at the end points.

   (c) To illustrate the method, construct the natural cubic spline (i.e. with $s''(x)$ being zero at the end points) which interpolates the function $f(x) = x^5 + 3x^2$ at the points $x = -1, 0, 1$.

3. A first-order ordinary differential equation with initial condition is given by

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t), \quad y(t_0) = y_0. \tag{1}$$

(a) Derive Euler's method for solving (1) numerically and show that it has local truncation error of $O(h^2)$. State its global truncation error. What do the local and global truncation errors measure?

(b) In terms of the stability of a numerical scheme for solving (1), define the growth factor g and show that, for a numerical scheme given by

$$y_{n+1} = \tau(y_n, t_n),$$

g is given by

$$g = \frac{\partial \tau}{\partial y_n}.$$

(c) Examining the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y, \tag{2}$$

define the region of absolute stability. Find and sketch the region of absolute stability for the Euler method. Is it A-stable?

(d) Consider now the backward Euler method, namely

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1}).$$

Find, for the differential equation (2), its growth factor g and show, by writing $\lambda h = a + ib$ or otherwise, that this method is A-stable.

4. (a) Prove that given $N+1$ distinct points there is a unique polynomial of degree N which passes through all of the points.

   (b) Find, using divided differences or otherwise, the polynomial of smallest degree which passes through the data points $(0,0)$, $(\frac{1}{2},1)$, $(1,1)$, $(\frac{3}{2},3)$. Also, find the polynomial of smallest degree which also passes through the point $(2,-2)$ in addition to the other four.

   (c) Given that the error e_N involved in interpolating a function $y(x)$ on the interval $[a,b]$ with a polynomial $P_N(x)$ of degree N, is

$$e_N(x) = y(\bar{x}) - P_N(\bar{x}) = \frac{y^{N+1}(\xi)}{(N+1)!}\prod_{j=0}^{N}(\bar{x}-x_j), \quad \bar{x} \in [a,b],$$

   where $\xi \in [a,b]$, estimate the number of interpolating points required to linearly interpolate the function $y(x) = xe^x$ on the interval $[0,1]$ such that $e_N < 10^{-6}$.

5. A first-order ordinary differential equation with initial condition is given by

$$\frac{dy}{dt} = f(y,t), \quad y(t_0) = y_0. \tag{3}$$

   (a) Write down the second-order Taylor's method for solving (3) numerically on an evenly spaced grid of step size h. What is the main limitation of using Taylor's method to solve the differential equation (3) and how do the Runge-Kutta methods improve on this?

   (b) Derive the second-order Runge-Kutta method for solving (3) and show that it can be written in the form

$$y_{n+1} = y_n + k_1 + \frac{k_2 - k_1}{2\alpha},$$

   where $k_1 = hf(y_n, t_n)$, $k_2 = hf(y_n + \alpha k_1, t_n + \alpha h)$, h is the step-size and α is a constant.

   (c) The Runge-Kutta methods are explicit and single step. Explain what is meant by these terms and write down examples of an implicit and of a multi-step method. Outline the advantages and disadvantages of these three types of method and describe how an implicit method can be implemented.