University of London

## EXAMINATION FOR INTERNAL STUDENTS

## For The Following Qualification:-

M.Sci.

## Mathematics 3706: Modular forms

COURSE CODE : MATH3706

UNIT VALUE $: 0.50$

DATE : 11-MAY-05

TIME : 10.00

TIME ALLOWED : 2 Hours

All questions may be attempted but only marks obtained on the best four solutions will count.
The use of an electronic calculator is not permitted in this examination.

Throughout this exam we shall use the following notation for a complex number $z=x+i y$ :

$$
\Re z=x, \quad \Im z=y, \quad \bar{z}=x-i y, \quad e(z)=\exp (2 \pi i z)
$$

1. (a) Show that every element of $\mathrm{SL}_{2}(\mathbb{R})$ may be expressed as a product of matrices of the form:

$$
\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right), \quad\left(\begin{array}{cc}
y & 0 \\
0 & y^{-1}
\end{array}\right), \quad\left(\begin{array}{cc}
1 & x \\
0 & 1
\end{array}\right), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^{\times} .
$$

(b) Let $\mu$ be the measure on the upper half plane $\mathcal{H}$ defined by

$$
d \mu(\tau)=\frac{1}{y^{2}} d x \wedge d y, \quad \tau=x+i y
$$

Show that $\mu$ is invariant under the action of $\mathrm{SL}_{2}(\mathbb{R})$ on $\mathcal{H}$.
(c) State without proof a fundamental domain $\mathcal{F}$ for $\mathrm{SL}_{2}(\mathbb{Z})$ in $\mathcal{H}$.

Show that $\mu(\mathcal{F})$ is finite.
2. (a) For $g=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \mathrm{SL}_{2}(\mathbb{R})$ and $\tau \in \mathcal{H}$ show that

$$
\Im(g \tau)=\frac{\Im(\tau)}{|c \tau+d|^{2}}
$$

(b) Let $\Gamma=\mathrm{SL}_{2}(\mathbb{Z})$. Define the spaces $M_{k}(\Gamma)$ and $S_{k}(\Gamma)$.

For $f, g \in M_{k}(\Gamma)$ show that the following function on $\mathcal{H}$ is $\Gamma$-invariant:

$$
\phi(\tau)=f(\tau) \overline{g(\tau)} \Im(\tau)^{k}
$$

Define the Petersson inner product $\langle f, g\rangle$ in the case that at least one of the forms $f$ and $g$ is a cusp form.
(c) Let $k \geq 4$ be even and let $E$ denote the weight $k$ level 1 holomorphic Eisenstein series, normalized as follows:

$$
E(\tau)=\sum_{\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \in \Gamma_{N} \backslash \Gamma}(c \tau+d)^{-k}, \quad \Gamma_{N}=\left\{ \pm\left(\begin{array}{cc}
1 & n \\
0 & 1
\end{array}\right): n \in \mathbb{Z}\right\} .
$$

Furthermore let $f$ be a weight $k$ level 1 cusp form.
(i) Show by "unfolding" the integral, that

$$
\langle f, E\rangle=\int_{\Gamma_{N} \backslash \mathcal{H}} f(\tau) \Im(\tau)^{k} d \mu(\tau)
$$

where $d \mu(\tau)=\frac{d x d y}{y^{2}}$ is the invariant measure on $\mathcal{H}$.
(ii) Hence by substituting the Fourier expansion of $f$ show that

$$
\langle f, E\rangle=0 .
$$

3. In what follows we shall normalize the weight $k$ action of $\mathrm{GL}_{2}(\mathbb{R})^{+}$on functions $f$ on $\mathcal{H}$ by

$$
\left.f\right|_{k}\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=(c \tau+d)^{-k} f\left(\frac{a \tau+b}{c \tau+d}\right)
$$

(a) Let $f(\tau)=\sum_{n=1}^{\infty} a(n) e(n \tau)$ be a weight $k$, level 1 cusp form, and let $p$ be a prime number. Suppose $\left(T_{p} f\right)(\tau)=\sum_{n=1}^{\infty} b(n) e(n \tau)$, where the Hecke operator $T_{p}$ is defined by

$$
T_{p} f=\left.f\right|_{k}\left(\begin{array}{ll}
p & 0 \\
0 & 1
\end{array}\right)+\left.\sum_{j=0}^{p-1} f\right|_{k}\left(\begin{array}{ll}
1 & j \\
0 & p
\end{array}\right)
$$

Show that

$$
b(n)=a(n / p)+p^{1-k} a(p n)
$$

where we are using the convention $a(n / p)=0$ for $n / p \notin \mathbb{N}$.
(b) Suppose that $f$ is a simultaneous eigenfunction for all the Hecke operators $T_{p}$ with eigenvalues $\lambda_{p}$, normalized so that $a(1)=1$.
(i) Show that $\lambda_{p}=p^{1-k} a(p)$.
(ii) Show that for all $n \in \mathbb{N}$,

$$
a(p n)=a(p) a(n)-p^{k-1} a(n / p)
$$

(iii) Let $m$ be coprime to $p$. Prove by induction on $r$ that

$$
a\left(p^{r} m\right)=a\left(p^{r}\right) a(m)
$$

(iv) Explain briefly how one obtains the Euler product expansion for $\Re_{s}$ sufficiently large:

$$
\sum_{n} a(n) n^{-s}=\prod_{p} \frac{1}{1-a(p) p^{-s}+p^{k-1-2 s}}
$$

4. In this question we shall write $E(\tau, s)$ for the real analytic Eisenstein series normalized as follows:

$$
E(\tau, s)=\pi^{-s} \Gamma(s) \sum_{(m, n) \in \mathbb{Z}^{2} \backslash\{0\}} \frac{\Im(\tau)^{s}}{|m \tau+n|^{2 s}}, \quad \tau \in \mathcal{H}, \quad \Re s>1
$$

We shall also write $\Theta(t)$ for the following theta series:

$$
\Theta(t)=\sum_{(m, n) \in \mathbb{Z}^{2}} \exp \left(-\frac{\pi|m \tau+n|^{2}}{y} t\right)
$$

You may assume without proof that $\Theta\left(t^{-1}\right)=t \Theta(t)$, and also that for large $t$ we have $\Theta(t)=1+O\left(t^{-N}\right)$ for every $N>0$.
(a) Show that for $\gamma \in \mathrm{SL}_{2}(\mathbb{Z})$,

$$
E(\gamma \tau, s)=E(\tau, s)
$$

(b) Prove the following integral representation of $E$ :

$$
E(\tau, s)=\int_{0}^{\infty}(\Theta(t)-1) t^{s} \frac{d t}{t}
$$

(c) Hence show that

$$
E(\tau, s)=\int_{1}^{\infty}(\Theta(t)-1)\left(t^{s}+t^{1-s}\right) \frac{d t}{t}-\frac{1}{s}-\frac{1}{1-s}
$$

(d) Deduce the meromorphic continuation and functional equation of $E(\tau, s)$.
5. (a) Let $k / \mathbb{Q}$ be a finite normal extension; let $\rho: \operatorname{Gal}(k / \mathbb{Q}) \rightarrow \mathrm{GL}_{n}(\mathbb{C})$ be a Galois representation, and let $p$ be a prime number which is unramified in $k$.
(i) Define the Frobenius element $\mathrm{Fr}_{p}$, and state to what extent it is unique.
(ii) Define the Artin $L$ factor $L_{p}(\rho, s)$.
(iii) Let $\chi$ be a primitive Dirichlet character modulo $N$. Write down a field extension $k / \mathbb{Q}$ and a representation $\rho$ of $\operatorname{Gal}(k / \mathbb{Q})$ such that

$$
\prod_{p \text { not dividing } N} L_{p}(\rho, s)=\sum_{n \text { coprime to } N} \chi(n) n^{-s} \quad(\Re s>1) .
$$

Describe the main steps in proving this result.
(b) Let $f(\tau)=\sum a(n) e(n \tau)$ and $g(\tau)=\sum b(n) e(n \tau)$ be weight $k$, level 1 Hecke eigenforms. Explain the main steps in proving the analytic continuation and functional equation of the following $L$-function:

$$
L(f \times g, s)=\sum_{n} a(n) b(n) n^{-s}
$$

Explain briefly how the analytic continuation and functional equation of $L(f \times g, s)$ are predicted by Langlands' functoriality principle.

