UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C311: Methods Of Mathematical Physics I

COURSE CODE	:	**MATHC311**
UNIT VALUE	:	**0.50**
DATE	:	**25-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Use the method of variation of parameters to show that

$$\tfrac{1}{4}x^2e^{-x}[2\ln x - 3]$$

is a particular integral of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}\ln x\,. \qquad (x > 0)$$

(b) Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 in the complex t-plane so that

$$\int_{C_i} e^{xt} f(t)\,dt \qquad (i = 1, 2)$$

are non-trivial independent solutions of the differential equation

$$x\frac{d^2y}{dx^2} + (2 - x)\frac{dy}{dx} - y = 0\,. \qquad (x \geq 0)$$

Determine two independent solutions in closed form and deduce that *only one* solution is bounded at $x = 0$.

2. When air resistance is ignored, the equation of motion of a simple pendulum is

$$\frac{d^2x}{dt^2} = -\sin x\,,$$

where x is the inclination of the pendulum to the downward vertical. Determine the nature of the singular points in the phase plane and show that the phase trajectories are given by

$$y^2 = 2\cos x + c\,,$$

where $y = dx/dt$ and c is a constant. Sketch the phase trajectories.

If initially $x = 0$ and $y = 2$, find the value of c and show that the pendulum swings up to the *horizontal* after time $\ln(1 + \sqrt{2})$.

3. It is known that when the constant ϵ is small compared with unity, the equation

$$\frac{d^2x}{d\theta^2} + \epsilon\frac{d}{d\theta}F(x) + \lambda(\epsilon)x = 0\,,$$

where $\lambda(\epsilon)$ is a constant which depends on ϵ and $F(x)$ is a regular function, possesses a 2π-periodic solution $x(\theta)$. By seeking a solution in the form:

$$\begin{aligned} x(\theta) &= x_0(\theta) + \epsilon\, x_1(\theta) + \epsilon^2 x_2(\theta) + \cdots\,, \\ \lambda(\epsilon) &= 1 + \epsilon\,\lambda_1 + \epsilon^2\lambda_2 + \cdots\,, \end{aligned}$$

where $x_n(\theta + 2\pi) = x_n(\theta)$ and $x_n'(0) = 0$, $(n = 0, 1, \ldots)$, obtain the differential equations for $x_0(\theta)$ and $x_1(\theta)$.

For the case when $F(x) = \frac{1}{3}x^3 - x$, find $x_0(\theta)$, if $x_0(0) > 0$, and show that

$$\lambda_1 = 0\,, \qquad x_1(\theta) = \sin^3\theta + A_1\,\cos\theta\,,$$

where A_1 is an undetermined constant. Describe, without detailed calculations, how the constants A_1 and λ_2 are determined.

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0\,,$$

where $\epsilon > 0$ is a constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\,\sin[t + \phi(t)]$ if

$$\begin{aligned} \dot{A} &= -\epsilon\, f(A\,\sin\chi, A\,\cos\chi)\,\cos\chi\,, \\ \dot{\phi} &= \epsilon\, A^{-1} f(A\,\sin\chi, A\,\cos\chi)\,\sin\chi\,, \end{aligned}$$

with $\chi = \phi + t$.

If $0 < \epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of Rayleigh's equation, $f(x, \dot{x}) = (\frac{1}{3}\dot{x}^2 - 1)\,\dot{x}$. If $A(0) = A_0 \geq 2$ and $\phi(0) = \phi_0$, show that $x(t)$ is given approximately by

$$x(t) = 2\left[1 + \left(\frac{4}{A_0{}^2} - 1\right)e^{-\epsilon t}\right]^{-1/2} \sin\left(t + \phi_0\right),$$

and deduce that there is a periodic solution for $x(t)$ which is also a limit cycle.

[You may assume that $\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$ and $\int_0^{2\pi} \cos^4\theta\, d\theta = \frac{3}{4}\pi$.]

5. State *without proof* a form of Watson's Lemma.

Throughout the interval $a \leq t \leq b$, the function $f(t)$ is continuous and the function $\phi(t)$ is twice-differentiable with a simple maximum at $t = t_0$, where $a < t_0 < b$. Show that as $x \to +\infty$,

$$\int_a^b e^{x\phi(t)} f(t)\, dt \sim e^{x\phi(t_0)} f(t_0) \left[\frac{2\pi}{x\,|\phi''(t_0)|}\right]^{\frac{1}{2}}.$$

How is this result modified when $t_0 = a$ or $t_0 = b$?

Hence, or otherwise, verify that as $x \to +\infty$,

$$(a) \qquad x! = \int_0^\infty t^x e^{-t}\, dt \sim \sqrt{(2\pi x)}\; x^x\, e^{-x},$$

$$(b) \qquad \int_0^{\pi/2} t^3 e^{x \sin t}\, dt \sim \frac{1}{8}\pi^3 e^x \sqrt{\frac{\pi}{2x}}.$$

[You may assume that $(-\frac{1}{2})! = \sqrt{\pi}$.]