UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C311: Methods Of Mathematical Physics I

COURSE CODE : **MATHC311**

UNIT VALUE : **0.50**

DATE : **18-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0915-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. Determine a function $f(t)$ of the complex variable t and contours C_1 and C_2 so that

$$\int_{C_i} e^{xt} f(t)\, dt \qquad (i = 1, 2)$$

are non-trivial independent solutions of the differential equation

$$3x\frac{d^2y}{dx^2} - (3x-2)\frac{dy}{dx} - y = 0 . \qquad (x \geq 0)$$

Show that *both* solutions are bounded at $x = 0$ and $x = +\infty$, but *only one* solution has a bounded derivative at $x = 0$. If $y(0) = 1$ for this solution, show that

$$y'(0) = \frac{1}{2} .$$

[You may assume that

$$\int_0^1 u^\lambda (1-u)^\mu \, du = \frac{(\lambda)!\,(\mu)!}{(\lambda+\mu+1)!} . \qquad (\lambda,\, \mu > -1)]$$

2. Explain what you understand by the terms *centre*, *node*, *spiral point* and *saddle point* in relation to the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} .$$

If this equation is derived from the equation of motion of a particle moving in a straight line, with $y = dx/dt$, show that the trajectory in the (x, y) plane is closed if and only if the motion of the particle is periodic.

The equation of motion of a particle is

$$\frac{d^2x}{dt^2} = x^2 - 1 .$$

Initially $x = 0$ and the speed of the particle is V_0. Investigate the nature of the singular points in the phase plane and give a sketch of the phase trajectories. Deduce that the motion of the particle is periodic if $|V_0| < 2/\sqrt{3}$. If $V_0 = 0$, show that the time period for the motion of the particle is

$$(2\sqrt{3})^{\frac{1}{2}} \int_0^{\pi/2} \operatorname{cosec}^{\frac{1}{2}}\theta \, d\theta .$$

3. Show that the differential equation

$$\frac{d^2x}{dt^2} + \epsilon f(x)\frac{dx}{dt} + g(x) = 0\,, \qquad (\epsilon > 0)$$

where $f(x)$ and $g(x)$ are integrable functions of x and ϵ is a constant, can be expressed in the form

$$\frac{dy}{dx} = \frac{g(x)}{\epsilon F(x) - y}\,,$$

where $F(x) = \int_0^x f(t)\,dt$, by means of the transformation

$$\frac{dx}{dt} = y - \epsilon F(x)\,.$$

Assuming that a unique periodic solution $x(t)$ exists with maximum value A and period T for all values of ϵ, show that for a certain closed curve γ in the (x, y) plane,

$$\oint_\gamma y f(x)\,dx = 0\,.$$

If $f(x) = \operatorname{sgn}(|x| - 1)$ and $g(x) = x$, show that if $\epsilon \ll 1$, then

$$\int_0^A (A^2 - x^2)^{\frac{1}{2}} f(x)\,dx = 0\,,$$

and deduce that $A = \sec(\pi/4 + \phi/2)$, where $\cos\phi = \phi$.

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0\,,$$

where ϵ is a positive constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\sin[t + \phi(t)]$ if

$$\begin{aligned} \dot{A} &= -\epsilon\, f(A\sin\chi, A\cos\chi)\cos\chi\,, \\ \dot{\phi} &= \epsilon\, A^{-1} f(A\sin\chi, A\cos\chi)\sin\chi\,, \end{aligned}$$

with $\chi = \phi + t$. If $\epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of Van der Pol's equation, $f(x, \dot{x}) = (x^2 - 1)\,\dot{x}$. If $A(0) = A_0 > 0$ and $\phi(0) = \phi_0$, show that $x(t)$ is given approximately by

$$x(t) = 2\,[1 - (1 - 4/{A_0}^2)e^{-\epsilon t}]^{-1/2}\sin(t + \phi_0)\,,$$

and deduce the limit cycle solution for $x(t)$.

[You may assume that $\int_0^{2\pi}\cos^2\theta\,d\theta = \pi$ and $\int_0^{2\pi}\sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\pi$.]

5. State *without proof* a form of Watson's Lemma. Throughout the interval $a \le t \le b$, the function $f(t)$ is continuous and the function $\phi(t)$ is twice-differentiable with a simple maximum at $t = t_0$, where $a < t_0 < b$. Show that as $x \to +\infty$,

$$\int_a^b e^{x\phi(t)} f(t)\, dt \sim e^{x\phi(t_0)} f(t_0) \left[\frac{2\pi}{x\, |\phi''(t_0)|} \right]^{\frac{1}{2}} .$$

How is this result modified when $t_0 = a$ or $t_0 = b$?

Hence, or otherwise, verify that as $x \to +\infty$,

$$\int_0^\infty t^x e^{-t}\, dt \sim \sqrt{(2\pi x)}\; x^x\, e^{-x} .$$

[You may assume that $(-\frac{1}{2})! = \sqrt{\pi}$.]