UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C311: Methods Of Mathematical Physics I

COURSE CODE	:	**MATHC311**
UNIT VALUE	:	**0.50**
DATE	:	**09-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0908-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. Show that the differential equation

$$x\frac{d^2y}{dx^2} + 8\frac{dy}{dx} - xy = 0\,, \qquad (x \geq 0)$$

has independent non-trivial solutions of the form

$$y_i(x) = \int_{C_i} e^{xt} f(t)\,dt\,, \qquad (i = 1, 2)$$

if the function $f(t)$ of the complex variable t and the contours C_i are suitably chosen.

Verify that the solutions which are *finite* at $x = 0$ are multiples of

$$\sum_{r=0}^{3} \frac{(-1)^r}{r!(3-r)!}\left(\frac{d}{dx}\right)^{2r}\left\{\frac{\sinh x}{x}\right\}.$$

2. Show that the second order differential equation

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right).$$

has a periodic solution $x(t)$ if and only if its phase trajectory is closed.

A particle moves in a straight line on a smooth horizontal plane with equation of motion

$$\frac{d^2x}{dt^2} + x - \epsilon^2 x^3 = 0\,,$$

where $\epsilon > 0$ is a constant. Sketch the phase trajectories.

If $x = 0$, $dx/dt = U > 0$ at $t = 0$, show that the motion of the particle is periodic if and only if $U < 1/\epsilon\sqrt{2}$.

If the motion is periodic, show that the amplitude of the motion is $\epsilon^{-1}(1-\alpha)^{\frac{1}{2}}$, where $\alpha^2 = 1 - 2\epsilon^2 U^2$, and that the period T of oscillations is given by

$$T = 4\sqrt{2}\int_0^{(1-\alpha)^{\frac{1}{2}}} \frac{dw}{[(w^2-1)^2 - \alpha^2]^{\frac{1}{2}}}.$$

3. The equation of motion of a particle is

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0\,,$$

where the functions $g(x)$ and $F(x) = \int_0^x f(u)\,du$ are odd regular functions of x. By defining $y = \dot{x} + F(x)$, show that

$$\oint_\Gamma F(x)\,dy = 0\,,$$

if Γ is a limit cycle trajectory in the (x, y) plane. Verify that if the point (x, y) lies on Γ, then so does $(-x, -y)$.

If $F(x)$ has a single zero at $x = \alpha > 0$, and $F(x) > 0$ for $x > \alpha$, show that the amplitude A (ie the maximum value of x) in the periodic motion of the particle is such that $A > \alpha$.

Taking $f(x) = \epsilon\,(x^2 - 1)$, $g(x) = x$, with the constant $\epsilon \gg 1$, and assuming that a unique periodic solution $x(t)$ exists, prove that in the periodic motion of the particle, the amplitude $A \approx 2$ and the period $T \approx [3 - \ln 4]\epsilon$.

4. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0\,,$$

where $\epsilon > 0$ is a constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\,\sin[t + \phi(t)]$ if

$$\begin{aligned} \dot{A} &= -\epsilon\, f(A\,\sin\chi, A\,\cos\chi)\,\cos\chi\,, \\ \dot{\phi} &= \epsilon\, A^{-1} f(A\,\sin\chi, A\,\cos\chi)\,\sin\chi\,, \end{aligned}$$

with $\chi = \phi + t$.

If $0 < \epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of Rayleigh's equation, $f(x, \dot{x}) = (\dot{x}^2 - 1)\,\dot{x}$. If $A(0) = A_0 > 0$ and $\phi(0) = \phi_0$, show that $x(t)$ is given approximately by

$$x(t) = \frac{2}{\sqrt{3}}\left[1 + \left(\frac{4}{3A_0{}^2} - 1\right)e^{-\epsilon t}\right]^{-1/2} \sin(t + \phi_0)\,,$$

and deduce the limit cycle solution for $x(t)$.

[You may assume that $\int_0^{2\pi} \cos^2\theta\,d\theta = \pi$ and $\int_0^{2\pi} \cos^4\theta\,d\theta = \frac{3}{4}\pi$.]

5. A function $F(t)$ can be expanded in a power series in t for $0 \leq t < t_0$ of the form

$$F(t) = \sum_{n=0}^{\infty} a_n t^{\lambda_n}, \qquad (a_0 \neq 0)$$

where the sequence $\{\lambda_n\}$ increases with n and $\lambda_0 > -1$. For $t > t_0$, the function satisfies the inequality

$$|F(t)| < Be^{ct}$$

for some constants B, c. Show that as $x \to +\infty$,

$$\int_0^{\infty} e^{-xt} F(t)\, dt \sim \sum_{n=0}^{\infty} \frac{a_n(\lambda_n)!}{x^{\lambda_n+1}}.$$

Hence obtain *two* leading terms in the asymptotic expansions of each of the following integrals as $x \to +\infty$:

$$(a) \quad \int_1^{\infty} e^{-(xt+t^2)}\, dt,$$

$$(b) \quad \int_0^1 t^x (1 - \ln t)^{1/2}\, dt.$$

[You may assume that $\int_0^{\infty} e^{-u} u^{\lambda}\, du = \lambda!$ $\qquad (\lambda > -1)$]