UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C311: Methods Of Mathematical Physics I

COURSE CODE	:	**MATHC311**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0906-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. Verify that the differential equation

$$x^5 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - 12\,y = 0 \qquad (x > 0)$$

does not possess a non-trivial solution in ascending powers of x.

By seeking a solution of the form

$$y = \exp\left[\int K(x)\,dx\right],$$

where $|K(x)| \to \infty$ as $x \to 0+$, obtain an *exact* solution of the differential equation which grows exponentially as $x \to 0+$.

Determine a second independent solution of the differential equation for y, leaving your solution in the form of an integral.

2. Explain what you understand by the terms *centre*, *node*, *spiral point* and *saddle point* in relation to the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

The equation of motion of a particle is

$$\ddot{x} = k\,(2\cos x - 1)\,\sin x\,,$$

where $k > 0$ is a constant. Identify and describe the singular points in the phase plane of x and $y(=\dot{x})$ and show that the equation of the phase trajectories is

$$y^2 = 2k\,[\sin^2 x + \cos x] + C\,,$$

where C is a constant. Sketch the phase trajectories and deduce that the particle motion is non-periodic if $C > 2k$.

3. Show that the equation

$$\ddot{x} + \epsilon f(x, \dot{x}) + x = 0\,,$$

where ϵ is a positive constant and the dot denotes differentiation with respect to t, possesses a solution of the form $x = A(t)\,\sin[t + \phi(t)]$ if

$$\begin{aligned} \dot{A} &= -\epsilon\, f(A\,\sin\chi, A\,\cos\chi)\,\cos\chi\,, \\ \dot{\phi} &= \epsilon\, A^{-1} f(A\,\sin\chi, A\,\cos\chi)\,\sin\chi\,, \end{aligned}$$

with $\chi = \phi + t$. If $\epsilon \ll 1$, describe a method for finding approximate solutions of these equations.

For the case of Van der Pol's equation, $f(x, \dot{x}) = (x^2 - 1)\,\dot{x}$. If $A(0) = A_0 > 0$ and $\phi(0) = \phi_0$, show that $x(t)$ is given approximately by

$$x(t) = 2\,[1 - (1 - 4/{A_0}^2)e^{-\epsilon t}]^{-1/2}\sin(t + \phi_0)\,,$$

and deduce the limit cycle solution for $x(t)$.

[You may assume that $\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$ and $\int_0^{2\pi} \sin^2\theta \cos^2\theta\, d\theta = \frac{1}{4}\pi$.]

4. Show that the differential equation

$$\frac{d^2x}{dt^2} + \epsilon f(x)\,\frac{dx}{dt} + g(x) = 0\,, \qquad (\epsilon > 0)$$

where $f(x)$ and $g(x)$ are integrable functions of x and ϵ is a constant, can be expressed in the form

$$\frac{dy}{dx} = \frac{g(x)}{\epsilon F(x) - y}\,,$$

where $F(x) = \int_0^x f(t)\,dt$, by means of the transformation

$$\frac{dx}{dt} = y - \epsilon F(x)\,.$$

Assuming that a unique periodic solution $x(t)$ exists with maximum value A and period T for all values of ϵ, show that for a certain closed curve γ in the (x, y) plane,

$$\oint_\gamma y f(x)\,dx = 0\,.$$

If $f(x) = \text{sgn}(|x| - 1)$ and $g(x) = x$, show that when $\epsilon \gg 1$,

$$A \approx 3, \qquad T \approx 2\epsilon \ln 3\,.$$

5. (a) State without proof a form of Watson's Lemma. Use the lemma to find *two* leading terms in the asymptotic expansions of each of the following integrals as $x \to +\infty$:

$$(i) \qquad \int_{-1}^{1} e^{-x(2t-t^2)}\, dt\,,$$

$$(ii) \qquad \int_{-1}^{1} e^{x(2t-t^2)} \sin \tfrac{1}{2}\pi(1-t)\, dt\,.$$

(b) Use the method of stationary phase to show that

$$\int_{-1}^{1} \cos\left[x(t-t^2)\right] \sin \tfrac{1}{2}\pi t\, dt \sim \left(\frac{\pi}{4x}\right)^{\frac{1}{2}} \left[\cos \tfrac{1}{4}x + \sin \tfrac{1}{4}x\right]$$

as $x \to +\infty$.