UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.*

Mathematics E003: Mathematics

COURSE CODE : **MATHE003**

UNIT VALUE : **0.50**

DATE : **19-MAY-06**

TIME : **10.00**

TIME ALLOWED : **3 Hours**

06-C1032-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator is *permitted in this examination.*

1. Find the residue of e^{bz}/z^{n+1} at its pole when n is a positive integer or zero and b is any real non-zero number.

   Evaluate
$$\oint \frac{e^{bz}}{z^{n+1}} dz$$
   around the unit circle $|z| = 1$. Deduce that
$$\int_0^{\pi} \exp(\cos\theta)\cos(\sin\theta - n\theta)d\theta = \pi/n!,$$
$$\int_0^{\pi} \cosh(\cos\theta)\cos(\sin\theta)d\theta = \pi.$$

2. (a) Show that, if
$$f \star g = \int_{-\infty}^{\infty} f(x-u)g(u)du,$$
   then $\mathcal{F}(f \star g) = \mathcal{F}(f).\mathcal{F}(g)$, where $\mathcal{F}(f)$ denotes the Fourier transform of $f(x)$.

   (b) Find the solution $g(x)$ of the integral equation
$$\int_{-\infty}^{\infty} \frac{g(u)du}{(x-u)^2 + a^2} = \frac{1}{x^2 + b^2},$$
   when $0 < a < b$.

   [You may assume that $\int_0^{\infty} (x^2 + \mu^2)^{-1} \cos(\lambda x)dx = \pi e^{-|\lambda|\mu}/(2\mu)$.]

3. The equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=\phi$$

is to be solved for $\phi(x,y)$ in the first quadrant $x \geqslant 0$, $y \geqslant 0$ of the (x,y) plane, with the boundary conditions:

(i) $\phi = e^{-x} \quad (y=0, x\geqslant 0)$,
(ii) $\phi = e^{-y} \quad (x=0, y\geqslant 0)$,

together with the condition $\phi \to 0$ as $x^2+y^2 \to \infty$.
Defining a Fourier sine transform

$$\hat{\phi}_S(x,k)=\int_0^\infty \phi(x,y)\sin ky\, dy,$$

verify that $\hat{\phi}_S(0,k)=k/(1+k^2)$. Show also that

$$\frac{\partial^2\hat{\phi}_S}{\partial x^2}-(1+k^2)\hat{\phi}_S=-ke^{-x}.$$

Solve this equation for $\hat{\phi}_S(x,k)$ and deduce that

$$\phi(x,y)=e^{-x}-\frac{2}{\pi}\int_0^\infty \frac{\sin ky \exp[-(1+k^2)^{\frac{1}{2}}x]}{k(1+k^2)}dk.$$

$\left[\text{You may assume that } \int_0^\infty \frac{\sin\theta}{\theta}d\theta=\frac{\pi}{2}.\right]$

4. Show that the Laplace transform of the function

$$g(t) = \begin{cases} t & (0\leqslant t\leqslant \frac{\pi}{2}),\\ \pi - t & (\frac{\pi}{2}\leqslant t\leqslant \pi),\\ 0 & (\text{other values of } t),\end{cases}$$

is

$$G(p)=\left(1-e^{-\frac{p\pi}{2}}\right)^2/p^2.$$

If a simple harmonic oscillator $x(t)$ satisfies $\ddot{x}+x=g(t)$, with $x(0)=\dot{x}(0)=0$, show that the solution for $t>0$ is

$$x(t)=\frac{1}{2\pi i}\int\frac{\left(1-e^{-\frac{p\pi}{2}}\right)^2 e^{pt}dp}{p^2(p^2+1)},$$

and indicate the path of integration in the p plane. Deduce that, for $t > \pi$, $x(t)=-2\cos t$.
Also, find $x(t)$ for $0\leqslant t<\frac{\pi}{2}$.

5. If $F(p)$, $G(p)$ are the Laplace transforms of $f(x)$, $g(x)$ respectively, show that the function $\int_0^x f(s)g(x-s)ds$ has the Laplace transform $F(p)G(p)$.
Consider the integral equation for the unknown function $f(x)$,

$$\int_0^x \frac{f(s)}{(x-s)^{1/2}} ds = h(x), \qquad x > 0,$$

where $h(x)$ is given and $h(0) = 0$.
By writing $g(x) = x^{-\frac{1}{2}}$, show that the Laplace transform of $f(x)$ is $F(p) = (p/\pi)^{1/2} H(p)$, where $H(p)$ is the transform of $h(x)$. Hence show that

$$f(x) = \frac{1}{\pi} \int_0^x (x-s)^{-\frac{1}{2}} \frac{dh(s)}{ds} ds.$$

[You may assume that $\int_0^\infty u^{-\frac{1}{2}} e^{-u} du = \pi^{\frac{1}{2}}$.]

6. (a) The functional

$$\int_a^b F(y, y')\mathrm{d}x$$

is minimised by $y = y(x)$ with $y(a)$ and $y(b)$ prescribed. Assuming that $y(x)$ satisfies Euler's equation, deduce that

$$F - y' \frac{\partial F}{\partial y'} = \text{constant}.$$

(b) A particle moves in the (x, y) plane from $(0,1)$ to $(2,1)$ along a smooth path with speed λy, where $\lambda > 0$ is a constant. Show that the time T taken for this motion is given by

$$T = \frac{1}{\lambda} \int_0^2 \frac{[1 + (y')^2]^{\frac{1}{2}}}{y} \mathrm{d}x\,.$$

If T is a minimum for this path, show that

$$y[1 + (y')^2]^{\frac{1}{2}} = \text{constant}.$$

Show that this can be rearranged as

$$\mathrm{d}x = y(\alpha^2 - y^2)^{-\frac{1}{2}} \mathrm{d}y$$

and hence show that the path is an arc of a circle of radius $\sqrt{2}$.

7. A smooth plane curve, of given length ℓ, is drawn through the fixed points with Cartesian coordinates $(0,0)$ and $(a,0)$, with $0 < a < \ell$, so that the area enclosed by the curve and the chord (i.e. the straight line through the fixed points) is a maximum.

(a) If the curve has equation $y = y(x)$, show that it is determininied by maximizing

$$I = \int_0^a y\mathrm{d}x,$$

subject to the constraint that

$$\int_0^a [1 + (y')^2]^{\frac{1}{2}}\mathrm{d}x = \ell,$$

and the end conditions $y(0) = y(a) = 0$.

(b) Show that y satisfies

$$\mathrm{d}x = (y-b)\mathrm{d}y/\sqrt{\lambda^2 - (y-b)^2},$$

for constants λ and b.

(c) Establish that the curve is a circular arc of radius c say, where

$$\frac{a}{2c} = \sin\left(\frac{\ell}{2c}\right).$$

You may find it convenient to use the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x/a) = 1/\sqrt{a^2 - x^2}.$$

8. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-xy}{x-y}.$$

(a) Find and sketch the curves in the (x,y) plane on which $\mathrm{d}y/\mathrm{d}x$ is zero or infinite.

(b) Hence indicate the regions in the (x,y) plane within which $\mathrm{d}y/\mathrm{d}x$ is positive and those where $\mathrm{d}y/\mathrm{d}x$ is negative.

(c) Show that the equation has two critical points, a saddle point at (1,1) and second critical point, and determine the nature of the second point.

(d) Two trajectories terminate and two trajectories start at (1,1). Find their slopes there, and sketch the trajectories near (1,1).

(e) Deduce that there is only one trajectory that passes through (1,1) and $(a,0)$ where $0 < a < 1$. Sketch this trajectory and those in its immediate neighbourhood.

(f) Sketch other representative trajectories, showing clearly where their slopes vanish or become infinite.

9. Show that the phase trajectories of the second order differential equation

$$\ddot{x} = \begin{cases} -1 & (x \geqslant 1), \\ 0 & (|x| < 1), \\ +1 & (x \leqslant -1), \end{cases}$$

are closed curves formed from the arcs of parabolas and straight lines. Hence, or otherwise, deduce that all solutions $x(t)$ are periodic functions of the time t (measured in seconds).

Find an expression for the time period T of $x(t)$ as a function of U, where $U > 0$ is the maximum value attained by $\dot{x}$. Show that the *least* possible value of T is 8 seconds.

[Note that the expression for T can be determined from the solution in the phase plane without actually solving the differential equation for $x(t)$.]

10. State, without proof, a form of Watson's Lemma and use it to show that as $x \to +\infty$

(a)

$$\int_0^{\pi/2} e^{-x\cos\theta}\,\mathrm{d}\theta \sim \frac{1}{x} + \frac{1}{x^3} + \frac{9}{x^5} + \mathcal{O}\left(\frac{1}{x^7}\right),$$

(b)

$$\int_0^{\pi/2} t^x \sin t\,\mathrm{d}t \sim \left(\frac{\pi}{2}\right)^{x+1}\left[\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3}\left(1 - \frac{\pi^2}{4}\right) + \mathcal{O}\left(\frac{1}{x^4}\right)\right].$$

[You may assume that $(-\frac{1}{2})! = \sqrt{\pi}$.]