UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.*

Mathematics E003: Mathematics

COURSE CODE	:	**MATHE003**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-05**
TIME	:	**14.30**
TIME ALLOWED	:	**3 Hours**

05-C0950-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. By considering integrals around suitably chosen contours in the complex plane, verify that:

   (a) $\displaystyle\int_0^{\pi} \frac{d\theta}{(5-3\cos\theta)} = \frac{\pi}{4},$ (b) $\displaystyle\int_0^{\infty} \frac{\cos x dx}{(a^2+x^2)^2} = \frac{\pi e^{-a}(1+a)}{4a^3}$ $(a>0)$.

2. If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, show that the Fourier transform of the convolution

$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

   is $F(k)G(k)$.
   If $f(x) = ae^{-a|x|}$, $g(x) = be^{-b|x|}$, with $a, b > 0$ constants and $a \neq b$, find $F(k)$ and $G(k)$. Hence, or otherwise, evaluate the convolution integral $h(x)$. Show that, as $a \to \infty$, $h(x) \to 2g(x)$.

3. Show that, if $\hat{g}(k)$ is the Fourier sine transform of $g(x)$, the Fourier sine transform of d^2g/dx^2 is $-k^2\hat{g}(k) + kg(0)$.
   The function $\phi(x,y)$ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \lambda^2\phi$$

   in the first quadrant (i.e. $x > 0, y > 0$) of the (x,y)-plane, and the boundary conditions $\phi(x,0) = Bx$ for $0 < x < C$, $\phi(x,0) = 0$ for $x > C$, $\phi(0,y) = 0$ for $y > 0$, with $\phi \to 0$ as $x, y \to \infty$. Here B, C are constants. Show that the Fourier sine transform of $\phi(x,y)$ with respect to x has the form

$$\hat{\phi}(k,y) = A(k)\exp\left[-\left(k^2+\lambda^2\right)^{1/2} y\right]$$

   and determine $A(k)$. Hence give an expression for $\phi(x,y)$ in terms of an integral.

4. Define the Laplace transform $\mathcal{L}\{y(x)\} = \bar{y}(p)$, and verify that

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} &= p^2\bar{y}(p) - py(0) - y'(0), \\
\mathcal{L}\left\{x\frac{dy}{dx}\right\} &= -\frac{d}{dp}[p\bar{y}(p)], \\
\mathcal{L}\left\{x\frac{d^2y}{dx^2}\right\} &= -\frac{d}{dp}[p^2\bar{y}(p)] + y(0).
\end{aligned}$$

Use these results to show that the equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0,$$

where λ is a constant, has a solution $y = Y_\lambda(x)$, such that

$$\mathcal{L}\{Y_\lambda(x)\} = \frac{(p-1)^\lambda}{p^{\lambda+1}}.$$

If $\lambda = n$, a positive integer, deduce that $Y_n(x)$ is a polynomial in x of degree n.

5. A rectangular pulse is defined by

$$f(t) = \begin{cases} 0, & (t < 0,\ t > n\pi) \\ 1, & (0 < t < n\pi) \end{cases}$$

with n a positive integer. Find the Laplace transform $\mathcal{L}\{f(t)\}$.

The function $x(t)$ satisfies the differential equation

$$\ddot{x} + x = f(t), \quad (t > 0)$$

with $x(0) = \dot{x}(0) = 0$ and the dot denoting differentiation with respect to time t. Show that

$$x(t) = \frac{1}{2\pi i}\int \frac{e^{pt}(1 - e^{-pn\pi})}{p(p^2+1)}dp,$$

and state the path of integration in the complex p-plane. Hence find $x(t)$ for $t > n\pi$ by evaluating the integral, and distinguish between the cases of odd and even values of n.

6. A heavy chain of constant density ρ per unit length hangs under gravity from two fixed points on the same horizontal level and distance a apart. The length of the chain is $l > a$.

(a) Show that

$$l = \int_0^a [1 + (y')^2]^{1/2} \mathrm{d}x$$

if the coordinates (x, y) are suitably chosen.

(b) Show also that the potential energy V of the chain is given, apart from a constant, by

$$V = -\rho g \int_0^a y[1 + (y')^2]^{1/2} \mathrm{d}x.$$

(c) Given that V is a minimum when the chain hangs in stable equilibrium, deduce that $y(x)$ satisfies

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \alpha^2(y + \beta)^2 - 1,$$

where α and β are constants.

(d) *Verify* that this equation and the conditions at $x = 0$ and $x = a$ are satisfied by

$$y = \alpha^{-1} \cosh[\alpha(x - \frac{1}{2}a)] - \beta,$$

when $\beta = \alpha^{-1} \cosh(\frac{1}{2}\alpha a)$.

(e) Show that α satisfies the equation

$$\sinh(\frac{1}{2}\alpha a) = \frac{1}{2}\alpha l.$$

7. The function $y(x)$ is required that minimises

$$I = \int_0^\pi [(y'')^2 + y^2]\mathrm{d}x,$$

subject to the end conditions $y(0) = y(\pi) = y''(0) = y''(\pi) = 0$ and the constraint

$$\int_0^\pi (y')^2\mathrm{d}x = \frac{1}{2}\pi.$$

By seeking a series solution with

$$y''(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

so that all the end conditions are satisfied, or otherwise, find the coeffiecents a_n and show that the minimum value of I is π.

You may assume that for $m, n = 1, 2, \ldots,$

$$\int_0^\pi \sin mx \sin nx \, dx = \int_0^\pi \cos mx \cos nx \, dx = \begin{cases} \frac{1}{2}\pi & (m = n) \\ 0 & (m \neq n). \end{cases}$$

8. Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - 1 + y}{y}.$$

(a) Find and sketch the curves in the (x, y) plane on which $\mathrm{d}y/\mathrm{d}x$ is zero or infinite.

(b) Hence indicate the regions in the (x, y) plane within which $\mathrm{d}y/\mathrm{d}x$ is positive and those where $\mathrm{d}y/\mathrm{d}x$ is negative.

(c) Show that the equation has two critical points, a saddle point at (1,0) and second critical point, and determine the nature of the second point.

(d) Two trajectories terminate and two trajectories start at (1,0). Find their slopes there, and sketch the trajectories near (1,0).

(e) Deduce that there is only one trajectory that passes through (1,0) and $(0, a)$ where $0 < a < 1$. Sketch this trajectory and those in its immediate neighbourhood.

9. Given that Van der Pol's equation

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0\,,$$

has a unique periodic solution with amplitude a, construct the periodic solution as a power series in ϵ when $0 < \epsilon \ll 1$. If $x(0) = a$ and $\dot{x}(0) = 0$, show that

$$a = 2 + a_1\,\epsilon + O(\epsilon^2)$$

and that the period of $x(t)$ is $2\pi/n$, where

$$n \;=\; 1 + n_2\,\epsilon^2 + O(\epsilon^3)\,.$$

Determine the constants a_1 and n_2.

10. State, without proof, a form of Watson's Lemma and use it to show that as $x \to +\infty$

(a)

$$\int_0^\infty e^{-xt^2} t^3 \log(1+t)\,\mathrm{d}t \sim \frac{3\pi^{1/2}}{8x^{5/2}} + \mathcal{O}\left(\frac{1}{x^3}\right),$$

(b)

$$\int_0^1 t^x (1-\log t)^{1/2}\,\mathrm{d}t \sim \frac{1}{x} - \frac{1}{2x^2} + \mathcal{O}\left(\frac{1}{x^3}\right).$$

[You may assume that $(-\frac{1}{2})! = \sqrt{\pi}$.]