UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.*

Mathematics E003: Mathematics

COURSE CODE	:	**MATHE003**
UNIT VALUE	:	**0.50**
DATE	:	**18-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

04-C0944-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. By considering contour integrals in the complex z-plane, with z suitably defined, show that

   (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{25 - 16\cos^2\theta} = \frac{2\pi}{15}$,

   (b) $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

2. From Cauchy's Residue Theorem, show that, in the domain of analyticity of a complex function $f(z)$, the integral

$$\int_C \frac{f(z)}{(z - z_1)} dz$$

(where C is a closed contour) is equal to

   (a) $2\pi i f(z_1)$ if C encloses the point z_1,

   (b) zero if z_1 is outside C.

Use the results (a), (b) to show that the solution of Laplace's equation for a real function $u(x, y)$, for $y > 0$, is

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{g(\xi)d\xi}{(x - \xi)^2 + y^2},$$

where $g(x) = u(x, 0)$ and $u \to 0$ as $x^2 + y^2 \to \infty$.
Show that, if

$$g(x) = \begin{cases} 1 & \text{for} \quad |x| < b, \\ 0 & \text{for} \quad |x| > b, \end{cases}$$

where b is a positive constant, then

$$\pi u(x, y) = \tan^{-1}\left(\frac{x+b}{y}\right) - \tan^{-1}\left(\frac{x-b}{y}\right).$$

3. The equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \phi$$

is to be solved for $\phi(x,y)$ in the first quadrant $x \geqslant 0$, $y \geqslant 0$ of the (x,y) plane, with the boundary conditions:

(i) $\quad \phi = e^{-x} \quad (y = 0, x \geqslant 0),$ $\qquad$ (ii) $\quad \phi = e^{-y} \quad (x = 0, y \geqslant 0),$

together with the condition $\phi \to 0$ as $x^2 + y^2 \to \infty$.

Defining a Fourier sine transform

$$\hat{\phi}_S(x,k) = \int_0^\infty \phi(x,y) \sin ky \, dy,$$

verify that $\hat{\phi}_S(0,k) = k/(1+k^2)$. Show also that

$$\frac{\partial^2 \hat{\phi}_S}{\partial x^2} - (1+k^2)\hat{\phi}_S = -ke^{-x}.$$

Solve this equation for $\hat{\phi}_S(x,k)$ and deduce that

$$\phi(x,y) = e^{-x} - \frac{2}{\pi}\int_0^\infty \frac{\sin ky \exp[-(1+k^2)^{1/2}x]}{k(1+k^2)} dk.$$

$\left[\text{You may assume that } \displaystyle\int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2}.\right]$

4. For a function $y(t)$ satisfying $y(0) = y'(0) = 0$ show that the Laplace transforms of $t\dfrac{\mathrm{d}y}{\mathrm{d}t}$ and $t\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}$ are $-\dfrac{\mathrm{d}(p\bar{y})}{\mathrm{d}p}$ and $-\dfrac{\mathrm{d}(p^2\bar{y})}{\mathrm{d}p}$, respectively, where $\bar{y}(p)$ is the transform of $y(t)$.

Hence, or otherwise, find a solution of the equation

$$t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - (3t+1)\frac{\mathrm{d}y}{\mathrm{d}t} + 3y = 0$$

such that $y(0) = y'(0) = 0$.

5. The current I flowing in a circuit with constant inductance L and resistance R satisfies the differential equation

$$L\frac{dI}{dt} + RI = E(t) \qquad (t > 0)$$

where the electromotive force $E(t)$is given by

$$E(t) = \begin{cases} E_0 & (0 < t < \pi), \\ -E_0 & (\pi < t < 2\pi), \end{cases}$$

$$E(t + 2\pi) = E(t),$$

with E_0 a constant. Show that if $\mathcal{L}$ denotes the Laplace transform then

$$\mathcal{L}\{E(t)\} = \frac{E_0}{p}\tanh\frac{\pi p}{2}.$$

If $I = 0$ when $t = 0$, show that

$$I(t) = \frac{E_0}{2\pi i L}\int_C \frac{e^{pt}\tanh\frac{1}{2}\pi p}{p(p + R/L)}dp$$

for a suitable contour C in the complex p-plane which should be defined. Use the residue theorem to deduce that

$$\begin{aligned} I(t) &= \frac{E_0}{R}\tanh\frac{\pi R}{2L}e^{-Rt/L} \\ &+ \frac{4E_0}{\pi}\sum_{n=0}^{\infty}\frac{[R\sin(2n+1)t - (2n+1)L\cos(2n+1)t]}{(2n+1)[R^2 + (2n+1)^2L^2]}. \end{aligned}$$

6. The function $y = \phi(t)$ is an extremal of the functional

$$I(y) = \frac{1}{2}\int_0^1 [(y')^2 + y^2]\mathrm{d}t,$$

and satisfies Euler's equation and the end conditions $y(0) = \alpha$, $y(1) = \beta$. Show that

$$I(\phi + f) \geqslant I(\phi),$$

for *any* twice differentiable function $f(t)$ vanishing at $t = 0$ and $t = 1$ and hence that $\phi(t)$ *minimizes* the functional I.

Determine the function $\phi(t)$ if $\alpha = 0$ and $\beta = 2$, and deduce that the minimum value that can be taken by I is $2(e^2 + 1)/(e^2 - 1)$.

7. (a) The functional

$$\int_a^b F(y, y')\mathrm{d}x$$

is minimised by $y = y(x)$ with $y(a)$ and $y(b)$ prescribed. Assuming that $y(x)$ satisfies Euler's equation, deduce that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

(b) The function $y = y(x)$ minimises the functional

$$I(y) = \int_{-1}^1 y^2(y')^2\mathrm{d}x,$$

and satisfies the end conditions $y(-1) = 0$, $y(1) = 0$ and the constraint

$$J(y) = \int_{-1}^1 y^2\mathrm{d}x = \frac{4}{3}.$$

Show that

$$[(y')^2 + \mu]y^2 = C,$$

where μ and C are constants, and deduce that the curve $y = y(x)$ is a semi-circle with centre at the origin $x = y = 0$.

8. Find the singular points of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 - 1}{x - y}$$

and determine their nature.

In what regions of the (x, y) plane is $\mathrm{d}y/\mathrm{d}x$ zero, positive, negative, or infinite? Use this information to sketch the trajectories of the differential equation.

9. The equation

$$\ddot{x} + \epsilon\dot{x}\,\mathrm{sgn}(|x| - 1) + x = 0,$$

where the constant $\epsilon \gg 1$, has a unique periodic solution $x(t)$. Show that the amplitude A and period T of $x(t)$ are given approximately by

$$A = 3, \qquad T = 2\epsilon \log 3.$$

10. State, without proof, a form of Watson's Lemma and use it to show that as $x \longrightarrow +\infty$

(a)

$$\int_x^\infty \frac{e^{-t}}{t}\mathrm{d}t \sim e^{-x}\sum_{n=0}^{\infty}(-1)^n \frac{n!}{x^{n+1}}.$$

(b)

$$\int_0^1 (1 - t^2)^x \mathrm{d}t \sim \sqrt{\pi}x^{-1/2} - \frac{3}{8}\sqrt{\pi}x^{-3/2} + \mathcal{O}\left(x^{-5/2}\right).$$

[You may assume that $(-\frac{1}{2})! = \sqrt{\pi}$.]