UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.* *M.Sc.*

Mathematics E003: Mathematics

COURSE CODE	:	**MATHE003**
UNIT VALUE	:	**0.50**
DATE	:	**12-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

03-C0937-3-50

TURN OVER

All *questions may be attempted but only marks obtained on the best* **seven** *solutions will count.*
The use of an electronic calculation **is** *permitted in this examination.*

1. Find the residue of the complex function

$$\frac{1}{(z^2+b^2)^2(z^2+c^2)}$$

at each of its poles in the upper half plane $\mathrm{Im}(z) > 0$, given that $b > 0,\ c > 0\ b \neq c$. Hence, or otherwise, show that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+b^2)^2(x^2+c^2)} = \frac{\pi(2b+c)}{2b^3c(b+c)^2}.$$

2. (a) Show that, if

$$f \star g = \int_{-\infty}^{\infty} f(x-u)g(u)du,$$

then $\mathcal{F}(f \star g) = \mathcal{F}(f).\mathcal{F}(g)$, where $\mathcal{F}(f)$ denotes the Fourier transform of $f(x)$.

(b) Find the solution $g(x)$ of the integral equation

$$\int_{-\infty}^{\infty} \frac{g(u)du}{(x-u)^2+a^2} = \frac{1}{x^2+b^2},$$

when $0 < a < b$.

[You may assume that $\displaystyle\int_0^{\infty} (x^2+\mu^2)^{-1}\cos(\lambda x)dx = \pi e^{-|\lambda|\mu}/(2\mu)$.]

3. The temperature distribution $T(x,t)$ in an infinitely long thin bar satisfies the heat-conduction equation

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \quad (-\infty < x < \infty,\ \ t \geqslant 0).$$

Initially the temperature distribution is $f(x)$ where $f(x) \to 0$ as $|x| \to \infty$. Assuming that $T \to 0$ as $|x| \to \infty$ for $t > 0$, and using a Fourier transform ($\mathcal{F}$) in x, show that

$$T(x,t) = \int_{-\infty}^{\infty} f(x-u)h(u,t)du,$$

where $\mathcal{F}\{h(x,t)\} = e^{-k^2 t}$. Determine $h(x,t)$ and hence deduce that, for the case when $f(x) = e^{-x^2}$,

$$T(x,t) = (1+4t)^{-\frac{1}{2}}\exp[-x^2/(4t+1)].$$

$\left[\text{You may assume that } \displaystyle\int_{-\infty}^{\infty} e^{-\alpha(x+\beta)^2}dx = \sqrt{\tfrac{\pi}{\alpha}}, \quad (\alpha > 0).\right]$

4. Show that the Laplace transform of the function

$$g(t) = \begin{cases} t & (0 \leqslant t \leqslant \frac{\pi}{2}), \\ \pi - t & (\frac{\pi}{2} \leqslant t \leqslant \pi), \\ 0 & \text{(other values of } t), \end{cases}$$

is

$$G(p) = \left(1 - e^{-\frac{p\pi}{2}}\right)^2 / p^2.$$

If a simple harmonic oscillator $x(t)$ satisfies $\ddot{x} + x = g(t)$, with $x(0) = \dot{x}(0) = 0$, show that the solution for $t > 0$ is

$$x(t) = \frac{1}{2\pi i} \int \frac{\left(1 - e^{-\frac{p\pi}{2}}\right)^2 e^{pt} dp}{p^2(p^2+1)},$$

and indicate the path of integration in the p plane. Deduce that, for $t > \pi$,

$$x(t) = -2\cos t.$$

Also, find $x(t)$ for $0 \leqslant t < \frac{\pi}{2}$.

5. Show that

(a) the Laplace transform of the function

$$\int_0^t f(s)h(t-s)ds$$

is $F(p)H(p)$ where $F(p)$, $H(p)$ are the transforms of $f(t)$, $h(t)$ respectively;

(b) the Laplace transforms of $\cos t$ and $\sin t$ are, respectively, $p/(p^2+1)$ and $1/(p^2+1)$;

(c) the Laplace transform of $\exp(at)f(t)$ is $F(p-a)$ if a is a constant.

Hence, or otherwise, solve the integral equation

$$g(t) - 2\int_0^t \cos(t-s)g(s)ds = \sin\left(t + \frac{\pi}{4}\right)$$

for the unknown function $g(t)$.

6. State, without proof, Euler's equation which is satisfied by an extremal of the functional

$$I(y) = \int_a^b F(x, y, y')\, dx\,,$$

with the values of $y(a)$ and $y(b)$ given.

Show that the functional

$$I(y) = \int_0^a [(y')^2 + y^2 + xy]\, dx\,, \qquad (a > 0)$$

with $y(0) = y(a) = 0$, has the extremal function

$$\phi(x) = \frac{a \sinh x - x \sinh a}{2 \sinh a}\,.$$

Obtain the identity

$$I(y) - I(\phi) = \int_0^a [(y' - \phi')^2 + (y - \phi)^2]\, dx\,,$$

where $y(x)$ is any function with a continuous derivative for $0 \le x \le a$ and $y(0) = y(a) = 0$. Hence deduce that I is minimized by $\phi(x)$.

7. A heavy chain of length ℓ has density ρ per unit length and hangs under gravity from two fixed points distance $2a$ apart on the same horizontal level. Write down an integral expression for the length ℓ of the chain if coordinates (x, y) are chosen with the x-axis horizontal and the origin at one end of the chain.

Given that the potential energy V of the chain can be written as

$$V = -\rho g \int_0^{2a} y[1 + (y')^2]^{\frac{1}{2}}\, dx\,,$$

where g is the (constant) acceleration due to gravity, and that V is a minimum when the chain hangs in stable equilibrium, deduce that $y(x)$ satisfies the differential equation

$$\left(\frac{dy}{dx}\right)^2 = [k^2(y + h)^2 - 1]\,,$$

where h and k are constants. Hence show that the stable equilibrium shape of the chain is a catenary with equation

$$y = -h + k^{-1} \cosh [k(x - a)]\,.$$

Give equations from which the constants h and k are determined.

8. The functions $x(t)$ and $y(t)$ satisfy the system of differential equations

$$\frac{dx}{dt} = y(x^2 + y^2) + x(4 - x^2 - y^2),$$

$$\frac{dy}{dt} = -x(x^2 + y^2) + y(4 - x^2 - y^2).$$

By using polar coordinates, or otherwise, show that the system has one limit cycle, which is stable, and that the time of description of one circuit of the limit cycle is $\frac{1}{2}\pi$.

Sketch the trajectories in the (x, y) plane, indicating the directions in which the trajectories are described with increasing t, and verify that $x = y = 0$ is an unstable node.

9. The equation of rectilinear motion of a particle, which is connected by a spring to a fixed point and which slides on a rough horizontal plane, is

$$\ddot{x} + \operatorname{sgn}(\dot{x}) + x = 0.$$

If $x = \dot{x} = 1$ when $t = 0$, show that the particle comes to rest permanently at $x = 3 - \sqrt{5}$.

Sketch the phase trajectory corresponding to the motion of the particle and show that the total time of its motion is $\pi + \cot^{-1} 2$.

10. State, without proof, Watson's Lemma and use it to show that as $x \longrightarrow +\infty$,

$$(a) \qquad \int_x^{\infty} \frac{e^{-t^2}}{t^2}\, dt \sim \frac{e^{-x^2}}{2x^3}\left[1 - \frac{3}{2x^2} + O\left(\frac{1}{x^4}\right)\right],$$

$$(b) \qquad \int_0^1 t^x (1 - \ln t)^{1/2}\, dt \sim \frac{1}{x} - \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right).$$