UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Eng. *M.Eng.*

Mathematics E003: Mathematics

COURSE CODE	:	**MATHE003**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**3 hours**

02-C0932-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. By considering integrals around suitably chosen contours in the complex plane, verify that:

(a) $\displaystyle\int_0^\pi \frac{d\theta}{(5-3\cos\theta)} = \frac{\pi}{4},$

(b) $\displaystyle\int_0^\infty \frac{\cos x dx}{(a^2+x^2)^2} = \frac{\pi e^{-a}(1+a)}{4a^3} \qquad (a>0).$

2. If $F(k)$ and $G(k)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, show that the Fourier transform of the convolution

$$h(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

is $F(k)G(k)$.

If $f(x) = ae^{-a|x|}, g(x) = be^{-b|x|}$, with constants $a, b > 0$ and $a \neq b$, find $F(k)$ and $G(k)$. Hence, or otherwise, evaluate the convolution integral $h(x)$. Show that, as $a \to \infty, h(x) \to 2g(x)$.

3. The equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \phi$$

is to be solved for $\phi(x, y)$ in the first quadrant $x \geqslant 0$, $y \geqslant 0$ of the (x, y) plane, with the boundary conditions:

(i) $\phi = e^{-x} \quad (y = 0, x \geqslant 0)$,

(ii) $\phi = e^{-y} \quad (x = 0, y \geqslant 0)$,

together with the condition $\phi \to 0$ as $x^2 + y^2 \to \infty$.

Defining a Fourier sine transform

$$\hat{\phi}_S(x, k) = \int_0^\infty \phi(x, y) \sin ky \, dy,$$

verify that $\hat{\phi}_S(0, k) = k/(1 + k^2)$. Show also that

$$\frac{\partial^2 \hat{\phi}_S}{\partial x^2} - (1 + k^2)\hat{\phi}_S = -ke^{-x}.$$

Solve this equation for $\hat{\phi}_S(x, k)$ and deduce that

$$\phi(x, y) = e^{-x} - \frac{2}{\pi} \int_0^\infty \frac{\sin ky \exp[-(1 + k^2)^{\frac{1}{2}} x]}{k(1 + k^2)} dk.$$

$\left[\text{You may assume that } \int_0^\infty \frac{\sin \theta}{\theta} d\theta = \frac{\pi}{2}.\right]$

4. Define the Laplace transform $\mathcal{L}\{y(x)\} = \bar{y}(p)$, and verify that

$$\begin{aligned}
\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} &= p^2\bar{y}(p) - py(0) - y'(0), \\
\mathcal{L}\left\{x\frac{dy}{dx}\right\} &= -\frac{d}{dp}[p\bar{y}(p)], \\
\mathcal{L}\left\{x\frac{d^2y}{dx^2}\right\} &= -\frac{d}{dp}[p^2\bar{y}(p)] + y(0).
\end{aligned}$$

Use these results to show that the equation

$$x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + \lambda y = 0,$$

where λ is a constant, has a solution $y = Y_\lambda(x)$, such that

$$\mathcal{L}\{Y_\lambda(x)\} = \frac{(p-1)^\lambda}{p^{\lambda+1}}.$$

If $\lambda = n$, a positive integer, deduce that $Y_n(x)$ is a polynomial in x of degree n.

5. If $F(p)$, $G(p)$ are the Laplace transforms of $f(x)$, $g(x)$ respectively, show that the function $\int_0^\infty f(s)g(x-s)ds$ has the Laplace transform $F(p)G(p)$.
Consider the integral equation for the unknown function $f(x)$,

$$\int_0^\infty \frac{f(s)}{(x-s)^{\frac{1}{2}}}ds = h(x), \qquad x > 0,$$

where $h(x)$ is given and $h(0) = 0$.
By writing $g(x) = x^{-\frac{1}{2}}$, show that the Laplace transform of $f(x)$ is $F(p) = (p/\pi)^{\frac{1}{2}}H(p)$, where $H(p)$ is the transform of $h(x)$. Hence show that

$$f(x) = \frac{1}{\pi}\int_0^\infty (x-s)^{-\frac{1}{2}}\frac{dh(s)}{ds}ds.$$

[You may assume that $\int_0^\infty u^{-\frac{1}{2}}e^{-u}du = \pi^{\frac{1}{2}}$.]

6. The functional

$$I(y) = \int_0^a F(y, y')\, dx$$

is minimized by $y = y(x)$ with $y(0)$ and $y(a)$ prescribed. If Euler's equation is satisfied, show that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}.$$

A curve $y = y(x)$ is drawn through the points with Cartesian coordinates $(0, c)$ and (a, b) and rotated through $360°$ about the x-axis. Show that the surface area of the figure generated is

$$2\pi \int_0^a y\,[1 + (y')^2]^{1/2}\, dx.$$

Find the curve $y = y(x)$ which gives the minimum surface area, and explain, without detailed calculations, how the constants appearing in your expression are determined.

7. A smooth plane curve, of given length ℓ, is drawn through the fixed points with Cartesian coordinates $(0, 0)$ and $(a, 0)$, with $0 < a < \ell$, so that the area enclosed by the curve and the chord (ie the straight line through the fixed points) is a maximum. If the curve has equation $y = y(x)$, show that the problem is to maximize

$$I = \int_0^a y\, dx\,,$$

subject to the constraint

$$\int_0^a [1 + (y')^2]^{1/2}\, dx = \ell\,,$$

and the end conditions $y(0) = y(a) = 0$.

Establish that the curve is a circular arc of radius c say, where

$$\frac{a}{2c} = \sin\left(\frac{\ell}{2c}\right).$$

8. The equation of motion of a pendulum, with damping proportional to the square of the angular velocity, is

$$\ddot{x} + c|\dot{x}|\dot{x} + k\sin x = 0\,,$$

where c, k are positive constants and x is the inclination of the pendulum to the downward vertical.

By writing $y = \dot{x}/\sqrt{k}$, obtain the differential equation in the (x, y) plane and show that when x *increases* with t, the equation of the phase trajectories is

$$y^2 = A\,e^{-2cx} + (1 + 4c^2)^{-1}[\,2\cos x - 4c\sin x\,]\,,$$

where A is a constant.

If initially the pendulum is given an angular velocity V while it is hanging in its stable equilibrium position, show that it will make at least one complete revolution before coming to rest if

$$V^2 > 2k(1 + 4c^2)^{-1}[\,e^{2c\pi} + 1\,]\,.$$

9. A particle is attached by a soft spring to a fixed point. It performs small amplitude oscillations with diplacement $x(t)$ and its equation of motion is

$$\ddot{x} + x = \epsilon x^3\,,$$

where $0 < \epsilon \ll 1$ is a constant. If $x = a$ and $\dot{x} = 0$ when $t = 0$, show that

$$x(t) \approx a\cos nt + \frac{1}{32}\epsilon a^3[\,\cos nt - \cos 3nt\,] + O(\epsilon^2),$$

where $n \approx 1 - \frac{3}{8}\epsilon a^2 + O(\epsilon^2)$.

[You may assume that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.]

10. State, without proof, Watson's Lemma and use it to show that as $x \to +\infty$,

(*a*) $$\int_0^1 e^{-x(1-t^2)}\,dt \sim \left[\frac{1}{2x} + \frac{1}{4x^2} + O\left(\frac{1}{x^3}\right)\right],$$

(*b*) $$\int_1^\infty e^{-(xt+t^2)}\,dt \sim e^{-(1+x)}\left[\frac{1}{x} - \frac{2}{x^2} + O\left(\frac{1}{x^3}\right)\right].$$

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