UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.* *M.Sc.*

Mathematics E002: Mathematics

COURSE CODE	:	**MATHE002**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-06**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

06-C1031-3-180

TURN OVER

All questions may be answered, but only marks obtained on the best **seven** *questions will count.*

The use of an electronic calculator **is** *permitted in this examination.*

1. (a) Let A be an $m \times n$ matrix and let B be a $p \times q$ matrix. Under what conditions on the numbers m, n, p, q is the product AB well-defined?

(b) Let A and B be matrices for which the product AB is well-defined. Express the transpose of AB in terms of A^{T} and B^{T}. Confirm that your formula holds when $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ by explicitly writing out the various transposes and performing the products.

(c) Define what we mean when we say that a square matrix is symmetric. Define also what we mean when we say that a square matrix is anti-symmetric (sometimes also called skew-symmetric). Write the matrix A in part (b) as the sum of a symmetric and an anti-symmetric matrix.

(d) Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be three-dimensional vectors, which may be considered as 3×1 matrices. Define the scalar product (also called dot product or inner product) of $\mathbf{a}$ and $\mathbf{b}$. Show also how this scalar product can be written in terms of a matrix multiplication.

(e) What is the definition of the inverse of a square matrix? If A and B are square matrices that each have an inverse, write down the inverse of their product AB in terms of A^{-} and B^{-1}, and prove the result.

(f) If $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 4 & -2 & -1 \\ -3 & -3 & -1 \\ 0 & 4 & -1 \end{pmatrix}$, then find the matrix B, explaining your procedure where necessary.

2. (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general 2×2 matrix. What is the determinant of A?

(b) Evaluate the determinant of the matrix $\begin{pmatrix} 2 & 2 & 2 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}$ in terms of the variables x, y and z. Write it in its simplest (factorized) form.

(c) Evaluate the determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0.5 & 3 & 3.2 \\ 0.25 & 9 & 10.24 \end{pmatrix}$. [*Hint:* Use (b).]

(d) Use a Gaussian elimination process to solve the following system of equations:

$$\begin{aligned} x_1 + 3x_2 - x_3 + x_4 &= 5 \\ 2x_1 + 8x_2 - x_3 + 3x_4 &= 17 \\ x_1 + 5x_2 + 5x_4 &= 21 \\ 3x_1 + 11x_2 - x_3 + 6x_4 &= 28 \end{aligned}$$

(e) Evaluate the determinant of the matrix of coefficients for the system studied in part (d). [*Hint:* Use the reduced matrix from your Gaussian elimination process.]

(f) When solving a set of linear equations, what would it mean if the determinant of the matrix of coefficients were to be very small compared to the coefficients themselves? Illustrate your answer by discussing the behaviour of the solution of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 3+\delta \end{pmatrix}$$

as a function of ϵ and δ, when ϵ and δ are very small positive or negative numbers.

3. (a) Define what we mean when we say that a matrix A has an eigenvalue λ with corresponding eigenvector $\mathbf{v}$.

(b) Describe briefly how you would find all the eigenvalue-eigenvector pairs for a given matrix A by use of the characteristic polynomial.

(c) Show that $\lambda = 2$ and $\lambda = -1$ are eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

(d) For the matrix in part (c), find a complete set of eigenvectors that satisfy the orthogonality condition $\mathbf{v}_i^T \mathbf{v}_j = 0$ for $i \neq j$.

4. (a) State what we mean when we say that a given function $f(x)$ is an even function. State also what we mean when we say that a given function $f(x)$ is an odd function.

(b) For each of the following functions, say whether it is even, odd, both or neither: (i) x^3, (ii) x^2, (iii) $x + x^2$, (iv) $|x|$, (v) 0.
Make sure to explain your answers by reference to the definitions that you gave in part (a).

(c) Suppose that f is an even function and g is an odd function. What can be said about fg?

(d) Suppose that f is an odd function. Compute $\int_{-L}^{L} f(x)\,dx$. Explain your reasoning.

(e) Recall that the Fourier series representation for a function $f(x)$ defined on an interval $(-L, L)$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

are the Fourier coefficients.

Sketch a graph of the function f that is periodic of period 2π and is defined by $f(x) = |x|$ for $-\pi < x \leq \pi$. Then find its Fourier series. [*Hint:* Use part (d) to simplify your calculation.]

5. Recall that if $f(t)$ is a function defined for all $t \geq 0$, then its Laplace transform $F(s)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

whenever this integral is convergent.

(a) Let a be an arbitrary real number. Show that the Laplace transform of the function $f(t) = e^{at}$ is given by $\dfrac{1}{s-a}$. Make sure to fully explain your answer and to state any restrictions that we are required to make on s.

(b) Show that $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$ and $\mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1}$. In doing so, you may use without proof the fact that the result from part (a) holds also for *complex* numbers a if suitable restrictions are placed on s.

(c) Let $f(t)$ be an arbitrary function, and let $F(s)$ be its Laplace transform. Show that $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, where dash denotes differentiation with respect to the variable t.

(d) Use Laplace transforms to solve the system of differential equations

$$\begin{aligned} \frac{dy}{dt} - z &= e^t \\ \frac{dz}{dt} - y &= -1 - e^t - 2\sin t \end{aligned}$$

with initial conditions $y(0) = 2$, $z(0) = 1$. Check your answer.

6. An object of mass m falls under gravity through a liquid whose viscosity is decreasing as time goes on (think of honey being heated), so that the frictional force on the object at time t is $-\alpha mv/(1+t)$, where v is the object's velocity and α is a positive constant. Let the velocity at time $t = 0$ be v_0.

   (a) Set up a differential equation for the object's velocity as a function of time. (Make sure you include *all* the forces acting on the object, including the earth's gravity! Recall that the gravitational force on an object of mass m is mg.)

   (b) Find the velocity as a function of time, by solving the differential equation from part (a) with the given initial condition.

7. (a) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 0 .$$

   Then find the particular solution corresponding to the initial condition $x(0) = x_0$ and $\frac{dx}{dt}(0) = v_0$.

   (b) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{\alpha t} ,$$

   assuming that $\alpha \neq -1$.

   (c) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = e^{-t} .$$

   (d) Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = \cos 2t .$$

8. (a) Let $f(x,y) = 3x^2 + 6xy + 2y^3 - \frac{x}{2} - \frac{y}{24}$. Find all the stationary points (also called critical points) of the function f. Then classify each one of them as either a local minimum, a local maximum, or a saddle point.

(b) The London Zoo has hired you to design an aquarium for their new marine exhibit. This aquarium is to have rectangular sides and bottom but no top, with a total volume of 64 cubic metres. The glass used on the vertical walls is four times as expensive (per unit area) as the concrete used on the bottom surface. Find the dimensions of the aquarium that attain the required volume at minimum total cost.

9. (a) Sketch the region $0 \le x \le 1 - y^2$ in the (x,y)-plane. Then find the integral of the function $f(x,y) = 2x + y^2$ over this region.

For the remainder of this problem, let $\mathbf{A}$ be the vector field defined by

$$\mathbf{A}(x,y,z) = \left(x - \frac{y^3}{3}\right)\hat{\mathbf{e}}_x + (x^2+1)\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$$

where $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are the Cartesian unit vectors (also called $\mathbf{i}, \mathbf{j}, \mathbf{k}$).

(b) Compute $\nabla \times \mathbf{A}$.

(c) Does there exist a scalar field $\varphi(x,y,z)$ such that $\mathbf{A} = \nabla\varphi$? If so, find one such scalar field; if not, explain why one does not exist.

(d) Let P_1 be the straight-line path from $(0,-1,0)$ to $(0,1,0)$, and let P_2 be the path from $(0,-1,0)$ to $(0,1,0)$ following the parabola $x = 1-y^2$, $z = 0$. Evaluate the integrals $\int_{P_1} \mathbf{A}\cdot d\mathbf{r}$ and $\int_{P_2} \mathbf{A}\cdot d\mathbf{r}$. Are they equal? If not, by how much do they differ? Explain how your answer to this last question follows alternatively from Stokes' theorem and the answer to part (a).

10. (a) Let $\varphi(x, \_\,, z) = e^{xy}\sin z$. Compute $\nabla\varphi$.

(b) Let $\mathbf{A} = \nabla\varphi$ be the vector field found in part (a). Compute $\nabla\times\mathbf{A}$. [*Hint:* There is an easy way.]

(c) Let $\mathbf{A} = \nabla\varphi$ be the vector field found in part (a). Let P be the curve parametrized by $x(t) = \cos t$, $y(t) = \sin t$, $z(t) = t$ for $0 \le t \le \pi/2$. Compute $\int_P \mathbf{A}\cdot d\mathbf{r}$. [*Hint:* There is an easy way.]

(d) Let $\mathbf{B}(x,y,z) = (x+e^y)\hat{\mathbf{e}}_x + (y^2+y)\hat{\mathbf{e}}_y - 2yz\hat{\mathbf{e}}_z$ where $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ are the Cartesian unit vectors (also called $\mathbf{i}, \mathbf{j}, \mathbf{k}$). Let S be the surface $x^2+y^2+z^2=4$. Compute $\int_S \mathbf{B}\cdot d\mathbf{n}$, where $d\mathbf{n}$ denotes the outward normal. [*Hint:* There is an easy way.]