UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.* *M.Sc.*

Mathematics E002: Mathematics

COURSE CODE	:	**MATHE002**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

05-C0949-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. (a) Find the determinants, $\det(A)$ and $\det(B)$, of the matrices A and B below

$$A = \begin{pmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and show that $\det(AB) = \det(A)\det(B)$.

(b) Using row operations, or otherwise, show that the value of the determinant

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 2\theta \end{vmatrix}$$

is $2(1+2\sin 2\theta)$.

(c) If D_n is the $n \times n$ determinant given by

$$\begin{vmatrix} a & b & & & & \\ b & a & b & & & \\ & b & a & b & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & b & a & b \\ & & & & b & a \end{vmatrix}$$

then, defining $D_0 = 1$ and $D_1 = a$, prove by induction that $D_n = aD_{n-1} - b^2 D_{n-2}$ and hence show that

$$D_5 = a(a^2 - b^2)(a^2 - 3b^2).$$

2. (a) Define the dot product of any two, general, three-dimensional vectors in terms of the angle between them, and hence demonstrate that if the vectors are orthogonal then their dot product is zero. Also write down an alternative definition for the dot product in matrix notation, again for any general three-dimensional vectors.

   (b) Define what we mean when we say that a matrix A has eigenvalues λ and eigenvectors $\mathbf{v}$ and briefly describe how you would find these for a given matrix by use of the characteristic polynomial.

   (c) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   and confirm that the eigenvectors are orthogonal.

3. (a) What do we mean when we say that a function is (i) even or (ii) odd?

   (b) Is the product of an even and an odd function even, or odd? Prove the result.

   (c) Show that if $f(x)$ is an odd function then

$$\int_{-L}^{L} f(x)\, dx = 0.$$

   (d) State, without proof, the general definition of the Fourier series representation for a function $f(x)$ defined on an interval $(-L, L)$, stating the requirements on the function $f(x)$ and giving the formulae for the Fourier coefficients.

   (e) Find a series of cosines of multiples of x that represents the following function $f(x)$ on an interval $(0, \pi)$

$$f(x) = \begin{cases} 0, & 0 \leqslant x < \pi/2 \\ \pi/2, & \pi/2 < x \leqslant \pi \end{cases}, \quad f(\pi/2) = \pi/4.$$

   (f) To what does the Fourier series converge at $x = 0$? Use the result to find a series representation for $\pi/4$.

   (g) To what does the Fourier series converge at $x = \pi/2$ and at $x = \pi$?

4. (a) Find all possible real solutions of the following system of equations (i)when $t = 1$ and (ii)when $t = 0$

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 1\,, \\ 3x_1 & - & x_2 & + & 2x_3 & = & 2\,, \\ x_1 & - & 5x_2 & & & = & t\,. \end{array}$$

(b) Use the Gauss-Jordan method to find the inverse of the matrix

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

and hence, or otherwise, solve the simultaneous equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 1\,, \\ x_1 & + & 2x_2 & - & x_3 & = & 0\,, \\ x_1 & + & x_2 & & & = & 1\,. \end{array}$$

(c) Briefly describe a standard Gaussian elimination technique for solving a system of equations written in the matrix form $A\mathbf{x} = \mathbf{b}$, for a general 3×3 matrix A and a non-zero vector $\mathbf{b}$. When would the Gauss-Jordan method (or any method to find the inverse of the matrix) be used in favour of a Gaussian elimination method?

5. (a) Find the stationary points of the function

$$f(x, y) = x^2y^2 + 2xy + y^2 + 4y$$

distinguishing between maxima, minima and saddle points.

(b) A rectangular box with a lid is to have a volume of 54 cubic units. The material for the base costs three times the material for the other five surfaces. Determine the dimensions of the box so that the total cost is a minimum.

6. The Laplace transform of the function $f(t)$ is defined as a unique function $F(s)$ given by

$$L\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\, dt = F(s).$$

(a) Given that $L\{\mathrm{e}^{at}\} = 1/(s-a)$ for a constant, and noting that $\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i} \sin t$ find the Laplace transforms of the functions $f(t) = \sin t$ and $f(t) = \cos t$.

(b) Given that $L\{f'(t)\} = sF(s) - f(0)$, find the expression for the Laplace transform of the function $f''(t)$ in terms of initial conditions. Here, a dash denotes differentiation with respect to the variable t.

(c) Hence use Laplace transforms to solve

$$\begin{array}{rcrcl} z''(t) & + & y'(t) & = & \cos t, \qquad z(0) = -1, \quad y(0) = 1, \\ y''(t) & - & z(t) & = & \sin t, \qquad z'(0) = -1, \quad y'(0) = 0. \end{array}$$

7. Sketch the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = ax$ in the xy-plane. Shade the region of the sketch over which the double integral

$$\int_{x=0}^{a} dx \int_{y=(ax-x^2)^{\frac{1}{2}}}^{(a^2-x^2)^{\frac{1}{2}}} f(x,y)\, dy$$

is taken. Use elementary geometry to find the area of the region.

(i) If $f(x,y) = 1$, evaluate the double integral by changing to polar coordinates, and show that the value is that of the area found above.

(ii) If $f(x,y) = (a^2 - x^2 - y^2)^{-\frac{1}{2}}$, use polar coordinates again to show that the value of the integral is a.

8. State Stokes' theorem for a vector field $\mathbf{F}(x,y,z)$ and a surface S bounded by a closed curve C.

Verify Stokes' theorem for the part of the surface $z = 4 - x^2 - y^2$ for which $z \geqslant 0$ when

$$\mathbf{F}(x,y,z) = (z + y, z - x, z^2).$$

9. If $\mathbf{F} = \text{grad}\,\phi$ show that

$$\int_C \mathbf{F}.\mathbf{dr} = [\phi]_C$$

where $[\phi]_C$ denotes the difference in ϕ at the end points of the curve C.

If

$$\mathbf{G_1} = \text{grad}((y+z)\exp(xz))$$

and

$$\mathbf{G_2} = (y+z)\mathbf{i} + (2x + y + 3z^2)\mathbf{j} + yz^2\mathbf{k}$$

evaluate both

$$\int \mathbf{G_1}.\mathbf{dr} \quad \text{and} \quad \int \mathbf{G_2}.\mathbf{dr}$$

from $(0,0,0)$ to $(1,1,2)$ along

(i) $\mathbf{r} = (t^3, t^2, 2t)$ with $0 \leqslant t \leqslant 1$

(ii) the straight line joining the two points.

10. The steady two-dimensional temperature field $T(x,y)$ in the slab bounded by the planes $x = 0, x = a, y = 0, y = b$ satisfies Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

The faces $x = 0, x = a$ and $y = 0$ are at zero temperature. Use the method of separation of variables to show that

$$T(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}$$

where B_n are constants.

If, in addition, the face $y = b$ has temperature

$$T(x,b) = A\sin\frac{3\pi x}{a},$$

where A is a constant, find the constants B_n. Give an expression for the heat transfer across the face $y = 0$.