UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.* *M.Sc.*

Mathematics E002: Mathematics

COURSE CODE	:	**MATHE002**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-04**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

04-C0943-3-150

TURN OVER

All *questions may be attempted but only marks obtained on the best* **seven** *solutions will count. The use of an electronic calculator* **is** *permitted in this examination.*

1. (a) When considering the simultaneous solution of a pair of straight lines in (x, y) space, show that there are 3 possible outcomes.

   (b) A set of simultaneous linear algebraic equations can be written in the matrix form $A\mathbf{x} = \mathbf{b}$, where A is a square matrix of coefficients. Explain what is meant by the term non-homogenous and state under what conditions a unique solution exists.

   (c) Using a standard Gaussian elimination process, solve the following set of equations

$$\begin{array}{rcrcrcr} 2x & + & 4y & + & 7z & = & 25 \\ 4x & + & 5y & + & 6z & = & 19 \\ 3x & + & y & - & 2z & = & -10 \end{array} .$$

   Check your solution and write down the value of the determinant of the matrix of coefficients.

   (d) Define the determinant of a general 3×3 matrix A in terms of its minors. Show that the determinant of the matrix formed by multiplying any row of A by a factor k is equal to k times the determinant of A. Hence or otherwise find the determinant of the following matrix

$$\begin{bmatrix} 4 & 8 & 14 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{bmatrix} .$$

   (e) If we consider two general vectors in 3-dimensional space define the vector product in terms of a determinant and hence show that $\mathbf{a} \times \mathbf{a} = 0$ for any vector $\mathbf{a}$.

2. State the divergence theorem for a vector field $\mathbf{F}$ and a volume V.

   Verify the divergence theorem when

$$\mathbf{F}(x, y, z) = (x^3,\ y^3 + 3x + z^2,\ x^2 + y^2),$$

   and V is the volume enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

3. (a) For a general matrix A, show that $(A^{-1})^{-1} = A$.

(b) Using an elimination technique, find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{bmatrix}.$$

(c) Confirm your result from part (b) using the cofactor method.

(d) What is the inverse of a general matrix whose only nonzero elements lie on the diagonal?

(e) Consider the matrix

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & x \end{bmatrix}$$

(i) For what values of x does $(I - D)^{-1}$ not exist?

(ii) Given that $x = 4$, write down $(I - D)$ and $(I - D)^{-1}$.

(iii) If $x = 4$ and $D(I + C) = C$ then find C.

4. (a) For a general $n \times n$ matrix A, define its eigenvalues and eigenvectors and briefly explain how you would find them.

(b) What is the relationship between the eigenvalues of A and those of A^2?

(c) Show that $\lambda = 4$ is the largest eigenvalue of the matrix

$$A = \begin{bmatrix} 5 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & -2 & 2 \end{bmatrix}$$

and hence find all its eigenvalues and the corresponding eigenvectors.

5. (a) State, without proof, the general formula for the Fourier series on $(-L, L)$ for a function $f(x)$, giving the equations for the Fourier coefficients a_n and b_n.

(b) Let $f(x) = x$ on $(0, L)$. Show that the half-range cosine series which represents $f(x)$ correctly can be written as

$$\frac{L}{2} - \frac{4L}{\pi^2} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos\left\{(2m+1)\frac{\pi x}{L}\right\}.$$

(c) Explain how you would find a Fourier sine series (you do not need to actually find the series) and discuss how this representation would differ from that found in part (b).

6. The Laplace transform of the function $f(t)$ is

$$L\{f(t)\} = F(s) \equiv \int_0^{\infty} e^{-st} f(t)dt.$$

(a) Find an expression for the Laplace transform of $f'(t)$, where the dash denotes differentiation with respect to the variable t.

(b) Find an expression for the Laplace transform of the function e^{at} , where a is a constant.

(c) Noting that $e^{i\theta} = \cos\theta + i\sin\theta$, find the Laplace transforms of the functions $\cos(at)$ and $\sin(at)$.

(d) Use Laplace transforms to solve the differential equation $\frac{dy}{dt} + 2y = \cos(t)$ where y is a function of the variable t and given that $y = 1$ when $t = 0$.

7. (a) State the sufficient condition for a function $f(x, y)$ to have a stationary point at (a, b), distinguishing between a maximum, a minimum and a saddle point.

(b) A function of the real variables x and y is given by $f(x, y) = (x^2 + y^2)^2 - 18(x^2 - y^2)$. Determine the nature of any fixed points.

(c) Use the results of part (b) to sketch the function f.

8. (a) Find the directional derivative of $V = (x+y)^2 \exp(z)$ at the point $(1,2,3)$ in the direction of the vector $\mathbf{A} = (1,1,1)$. In which direction is the magnitude of the derivative at the point (1,2,3) greatest?

(b) Find $\int \mathbf{G}.\mathbf{dr}$ for $\mathbf{G} = (x^2y, zy, zx)$ from $(0,0,0)$ to $(1,1,1)$ along

(i) $\mathbf{r} = (t, t^2, t^3)$ with $0 \leqslant t \leqslant 1$,

(ii) the straight line joining the two points.

Is $\mathbf{G}$ conservative?

9. (a) Define the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)}.$$

(b) Determine the Jacobian, from your definition, for the change of coordinates from Cartesian coordinates to plane polar coordinates.

(c) By a suitable change of coordinates, or otherwise, find

$$I = \int_0^\infty \int_0^\infty (x^2+y^2)^{\frac{3}{2}} \exp\left(-\left(x^2+y^2\right)^{\frac{5}{2}}\right) dx\, dy.$$

(d) By reversing the order of integration, find

$$J = \int_0^1 \int_y^1 x \exp(x^3) dx\, dy,$$

showing in a diagram the region of integration.

10. The equation for the displacement $y(x,t)$ of a vibrating string stretched between two points on the x-axis is given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

where c is a constant. Describe the physical problem to which the following conditions apply:

$$y(0,t) = 0, \quad y(a,t) = 0, \quad y(x,0) = 0, \quad \partial y(x,0)/\partial t = f(x).$$

Use the method of separation of variables to solve this problem and show that, for $t > 0$ and $0 < x < a$,

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{a} \sin\frac{n\pi ct}{a}.$$

Give a formula, in the form of an integral involving the function f, for the constants b_n.