UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.*

Mathematics E002: Mathematics

COURSE CODE	:	**MATHE002**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-03**
TIME	:	**10.00**
TIME ALLOWED	:	**3 Hours**

03-C0936-3-120

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator is *permitted in this examination.*

1. (a) Find the general solution of

$$\begin{array}{rcrcrcrcl} x & + & 2y & - & 3z & + & 2w & = & 2 \\ 2x & + & 5y & - & 8z & + & 6w & = & 5 \\ 3x & + & 4y & - & 5z & + & 2w & = & 4 \\ x & + & y & - & z & - & 2w & = & 1 \end{array}$$

Verify that your solution is correct.

(b) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$$

Verify that your solution is correct.

2. (a) Define the eigenvalues and eigenvectors of a square matrix A.

(b) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}$$

You may find it useful to observe at some stage that one of the eigenvalues is equal to two.

(c) Two $n \times n$ matrices A and B each have n distinct eigenvalues and n distinct eigenvectors. Show that if $AB = BA$ then the eigenvectors of A and B correspond.

3. (a) State without proof the general formula for the Fourier series on $(-L, L)$ for a function $f(x)$, giving the equations for the Fourier coefficients a_n and b_n.

(b) Let $f(x) = x(L - x)$ on $(0, L)$. Show that the half-range cosine series that converges to $f(x)$ on $(0, L)$ can be written as

$$f(x) = \frac{1}{6}L^2 - \frac{4L^2}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos[(2k)\pi x/L]}{(2k)^2}.$$

4. The Laplace transform of a function $y(t)$ is $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st}y(t)\ \mathrm{d}t$.

(a) Show that

$$\mathcal{L}\left\{\frac{\mathrm{d}y}{\mathrm{d}t}\right\} = sY(s) - y(0),$$
$$\mathcal{L}\{t\} = 1/s^2,$$
$$\mathcal{L}\{e^{at}y(t)\} = Y(s-a).$$

(b) Use Laplace transforms to solve the following system of equations

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} + 4y - z &= 0,\\ \frac{\mathrm{d}z}{\mathrm{d}t} + 4y \qquad &= 0,\\ y(0) = 0, z(0) &= 1.\end{aligned}$$

5. State sufficient conditions for a function $f(x, y)$ to have a stationary point at (a, b) distinguishing between a maximum, a minimum and a saddle point.

Show that the function

$$f(x, y) = x^3 + 2x^2 - 3xy + y^2$$

has just one saddle point, S (say) and find the form of the other stationary point. Write down the form of $f(x, y)$ in the neighbourhood of S and hence obtain the equations of the two straight contours (lines along which $f(x, y) =$ constant) passing through S. Sketch the contour lines in the neighbourhood of S.

6. (a) Sketch the region over which the double integral

$$\int_{x=0}^{1}\int_{y=x^2}^{1} xy(x^2+y^2)\ \mathrm{d}y\ \mathrm{d}x$$

is taken. Interchange the order of the integration and hence evaluate the integral.

(b) Show that the ellipsoid $x^2 + y^2 + 3z^2 = 5$ and the parabolic cylinder $y^2 = z$ both pass through the point P with coordinates $(1, 1, 1)$. Find the equation of the normal line to the ellipsoid at P, and the equation of the normal line to the cylinder at P. Find also the equation of the plane that contains these two lines.

7. Show that

$$\boldsymbol{F} = (yz + y\cosh yz, xz + x\cosh yz + xyz\sinh yz, xy + xy^2\sinh yz)$$

is a conservative field of force, and find the work done by $\boldsymbol{F}$ in the displacement of a particle along any path from (0,1,2) to (1,2,3).

If $\boldsymbol{G} = (yz + y\cosh yz, xz + x\cosh yz + xyz\sinh yz, xy^2\sinh yz)$
find the values of

$$\int_C \boldsymbol{G}.\,\mathrm{d}\boldsymbol{r}$$

where C is that part of the curve $x = t,\ y = t^2 + 1,\ z = t^3 + 2$ between the points (0,1,2) and (1,2,3).

8. If $\phi(x, y, z)$ is a scalar field and $\boldsymbol{E}(x, y, z)$ is a vector field show that

$$\operatorname{div}\phi\boldsymbol{E} = \phi\operatorname{div}\boldsymbol{E} + \boldsymbol{E}.\operatorname{grad}\phi.$$

Write down the divergence theorem for the vector field $\boldsymbol{F}(x, y, z)$ and take $\boldsymbol{F} = \phi\operatorname{grad}\phi$ to show that

$$\int_V (\phi\operatorname{div}\operatorname{grad}\phi + |\operatorname{grad}\phi|^2)\,\mathrm{d}V = \int_S \phi\hat{\boldsymbol{n}}.\operatorname{grad}\phi\,\mathrm{d}S$$

where $\hat{\boldsymbol{n}}$ is the outward unit normal to the surface S enclosing the volume V.

If $\phi(x, y, z) = xy$ evaluate *either* the volume integral *or* the surface integral when V is the volume enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

9. State Stokes' theorem for a vector field $\boldsymbol{F}(x, y, z)$ and a surface $\boldsymbol{S}$ bounded by a curve C.

Verify the theorem when $\boldsymbol{F}(x, y, z) = xz\boldsymbol{i} - xy\boldsymbol{k}$, and the open surface $\boldsymbol{S}$ is constructed from the tetrahedron formed by the coordinate planes and the plane $x + y + z = 1$ by removing the face in $y = 0$.

10. The temperature $\theta(x, t)$ in a copper bar of length l satisfies the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{k}\frac{\partial\theta}{\partial t}, \qquad k \quad \text{constant.}$$

Solve the equation by separating the variables if the initial temperature distribution is $\theta_0\sin(n\pi x/l)$, where θ_0 is a constant and n is a positive integer, and the ends of the bar are maintained at 0°C.

If $\theta_0 = 100$°C, $l = 80$ cm, $n = 1$ and k=1.158 cm^2/sec, how long will it take for the maximum temperature in the bar to drop to 50°C?