UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Eng. *M.Eng.*

Mathematics E002: Mathematics

COURSE CODE	:	**MATHE002**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-02**
TIME	:	**10.00**
TIME ALLOWED	:	**3 hours**

02-C0931-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. (a) Find the general solution of

$$\begin{array}{rcrcrcrcr} 2x & + & 4y & + & 6z & + & 8w & = & -16 \\ -3x & + & 5y & - & 2z & + & 2w & = & 11 \\ 2x & + & 4y & + & 2z & + & 4w & = & -4 \\ 3x & - & 3y & - & 2z & + & 4w & = & -15 \end{array}$$

Verify that your solution is correct.

(b) Evaluate the determinant

$$\begin{vmatrix} 4 & -1 & 0 & 3 \\ -2 & 0 & 1 & -2 \\ -2 & 3 & 0 & 2 \\ 3 & -1 & -2 & 2 \end{vmatrix}.$$

2. (a) Define the eigenvalues and eigenvectors of a square matrix A.

(b) Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

(c) For a general $n \times n$ matrix B, show that the matrix B^2 has the same eigenvectors. What are the eigenvalues of B^2? Justify your answer.

3. (a.) Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & -3 \\ 2 & 2 & 3 \end{pmatrix}.$$

Verify your solution.

(b) Hence or otherwise solve the system of equations

$$\begin{aligned} x &- 2y + z = -2 \\ -2x &+ 2y - 3z = 1 \\ 2x &+ 2y + 3z = 3 \end{aligned}$$

(c) Let A be an $n \times m$ matrix and B an $m \times p$ matrix. Show that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

4. (a) State without proof the general formula for the Fourier series on $(-L, L)$ for a function $f(x)$, giving the equations for the Fourier coefficients a_n, b_n.

(b) Let $f(x) = x$ on $(0, L)$. Show that the half-range cosine series which represents $f(x)$ correctly can be written as

$$\frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\left[(2n+1)\frac{\pi x}{L}\right].$$

5. The Laplace transform of a function $y(t)$ is $\mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty e^{-st}y(t)dt$.

(a) Show that $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$ and $\mathcal{L}\{e^{at}y(t)\} = Y(s-a)$.

(b) Show that $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ and $\mathcal{L}\{e^{at}\sin at\} = \dfrac{a}{s^2-2sa+2a^2}$.

(c) Use Laplace transforms to solve the following system of equations:

$$\begin{aligned} \frac{dy}{dt} - z &= 0 \\ \frac{dz}{dt} - 4y &= 1 \\ y(0) = z(0) &= 0. \end{aligned}$$

6. (i) Define the directional derivative at a point P of the function $f(x, y, z)$ in the direction of the unit vector $\hat{\mathbf{t}}$. In what direction is the magnitude of this derivative a maximum? Find this maximum value if

$$f(x, y, z) = x^2 + 3y^2 + z^2 + xy + 2yz + 3zx$$

and P is the point $(1, 2, -3)$.

(ii) The double integral

$$\iint \exp\left(\frac{x-y}{x+y}\right) dxdy$$

is taken over the triangle enclosed by $x = 0$, $y = 0$, $x + y = 1$. Make the change of variables $u = x - y$, $v = x + y$ to show that its value is $\frac{1}{4}(e - e^{-1})$.

7. State sufficient conditions for the function $f(x, y)$ to have a stationary point at (a, b) distinguishing between a maximum, a minimum and a saddle point.

Show that the function

$$f(x, y) = (x^2 + y^2)^2 - 8(x^2 - y^2)$$

has two minima and one saddle point. Write down the form of $f(x, y)$ in the neighbourhood of the saddle point and hence give a sketch of the curves $f(x, y) =$ constant in this neighbourhood.

8. Let $\mathbf{F}$ denote the force field

$$\mathbf{F} = (3x^2 + yz)\mathbf{i} + (6y^2 + zx)\mathbf{j} + (12z^2 + xy)\mathbf{k}.$$

Show that curl $\mathbf{F} = 0$ and find a scalar potential φ such that $\mathbf{F} = \text{grad}\varphi$. Hence evaluate the integral

$$\int \mathbf{F}.d\mathbf{r}$$

from the point $(4, 2, -1)$ to the point $(1, -2, 1)$.

If

$$\mathbf{G} = (3x^2 + 2yz)\mathbf{i} + (6y^2 + zx)\mathbf{j} + (12z^2 + xy)\mathbf{k}$$

evaluate

$$\int \mathbf{G}.d\mathbf{r}$$

along the straight line joining the two points.

9. State the divergence theorem and use it to show that

$$\iint_S \varphi\hat{\mathbf{n}}dS = \iiint_V \text{grad}\varphi dV$$

where $\varphi(x, y, z)$ is a scalar function and $\hat{\mathbf{n}}$ is the outward unit normal to the surface S that encloses the volume V.

If $\varphi = x^2 + y^2 + z^2$ and the volume V is bounded by the paraboloid $z = a^2 - x^2 - y^2$ and the plane $z = 0$, evaluate either the surface or the volume integral given above.

10. The temperature $\theta(x, t)$ along a thin bar of length L satisfies the equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}, \qquad 0 < x < L, \quad 0 < t,$$

where t denotes time and x is the distance from one end.

The bar is insulated at each end so that $\partial\theta/\partial x = 0$ when $x = 0$ and $x = L$, and the initial temperature distribution is

$$\theta(x, 0) = 2\theta_0 \cos^2 \frac{\pi x}{L}$$

where θ_0 is a constant. Find $\theta(x, t)$ and show that the temperature throughout the bar tends to the uniform value θ_0.

END OF PAPER