UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. *M.Eng.*

Mathematics E001: Mathematics

COURSE CODE : **MATHE001**

UNIT VALUE : **0.50**

DATE : **05-MAY-04**

TIME : **10.00**

TIME ALLOWED : **3 Hours**

04-C0942-3-190

TURN OVER

All *questions may be attempted but only marks obtained on the best* **seven** *solutions will count. The use of an electronic calculator* **is** *permitted in this examination.*

1. (a) Find the modulus and argument of $1 - i\sqrt{3}$. Hence find the square roots of $1 - i\sqrt{3}$.

   (b) Use complex numbers to find

$$\int e^x \sin x \, dx.$$

   Show, also using complex numbers, that

$$\frac{d^n}{dx^n}(e^x \sin x) = 2^{n/2} e^x \sin\left(x + \frac{n\pi}{4}\right).$$

2. Define $\cosh x$ in terms of exponential functions and sketch the curve $y = \cosh x$. Show that

$$y - \sqrt{y^2 - 1} = \frac{1}{y + \sqrt{y^2 - 1}},$$

   and use this to show that

$$\cosh^{-1} y = \pm \ln\left(y + \sqrt{y^2 - 1}\right).$$

   By referring to your sketch of $y = \cosh x$, explain why there are both positive and negative values for $\cosh^{-1} y$.

   Use integration by parts to show

$$\int_1^2 \cosh^{-1} y \, dy = 2\ln(2 + \sqrt{3}) - \sqrt{3}.$$

3. (a) Differentiate

(i) $3^{\cos x}$ (ii) $\ln\left(\cos\left(\frac{1}{x^2}\right)\right)$.

(b) Find the stationary points of the function $y = x^2 e^{-x}$ and determine whether they are maxima, minima or points of inflection.

(c) If $x = a(\cos t + t\ \sin t)$ and $y = a(\sin t - t\ \cos t)$, where a is a constant, show

$$\frac{dy}{dx} = \tan t.$$

Find also

$$\frac{d^2y}{dx^2}.$$

4. (a) Find the following integrals

(i) $\int \tanh x\ dx$,

(ii) $\int_1^2 \frac{dx}{(4+x)\sqrt{x-1}}$,

(iii) $\int \frac{3-2x}{\sqrt{4x-x^2-3}} dx$.

(b) Show that

$$\int \sec x\ dx = \int \frac{\cos x}{1-\sin^2 x} dx. \qquad (*)$$

Use the substitution $\sin x = t$ on the RHS of (*) to show

$$\int \sec x\ dx = \ln|\sec x + \tan x| + C.$$

5. (a) Define carefully the dot and vector products of two vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) Two unit vectors $\hat{\mathbf{c}}$ and $\hat{\mathbf{d}}$ are perpendicular. Find $(\hat{\mathbf{c}} \times \hat{\mathbf{d}}) \times \hat{\mathbf{c}}$.

(c) The three points (4, −3, 2), (1, 0, 1) and (1, −1, 2) form the vertices of a triangle. Use vector methods to find the angle between the two sides of the triangle which meet at (1, 0, 1). Find, also, using vector methods, the area of the triangle.

6. (a) Let $f(x, y) = xy \sin\left(\frac{x}{y}\right)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Verify that $f(x, y) = \exp(-(1 + a^2)x) \cos(ay)$ is a solution of the equation

$$\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y^2} - f$$

for any constant a.

(c) Let $z = yf(xy)$ where f is any differentiable function. Show that

$$y\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial x} = z.$$

7. Solve the following differential equations:

(a) $\frac{dy}{dx} = 2x(1 + y^2), \qquad y(0) = 0;$

(b) $\frac{dy}{dx} + \frac{1}{x}y = e^x, \qquad y(1) = 1.$

8. (a) Find the general solution of the differential equation

$$y'' + y' + 4y = e^{4x}.$$

(b) Solve the following initial-value problem

$$y'' - 16y = 8x^2, \qquad y(0) = 0, \quad y'(0) = 1.$$

9. (a) Find the following limits:

(i)

$$\lim_{x \to 0} \frac{1 - \cos(x^2)}{2x^4};$$

(ii)

$$\lim_{n \to \infty} \frac{(3^n + n^2)^2}{n^4 + 3^{2n+3}}.$$

(b) Determine whether the following series are convergent or divergent, justifying your answer.

(i)

$$\sum_{n=1}^{\infty} \frac{n^2 + n + 1}{n^4 + 4n};$$

(ii)

$$\sum_{n=0}^{\infty} \frac{n^2 + 1}{2^n + 1}.$$

10. (a) Use Newton's method to find an approximate solution of the equation $e^x = 3x$ to three decimal places, starting from $x = 0$.

(b) Define the Poisson probability distribution with mean μ.

(c) On average, 0.2% of the chairs made at a factory are defective. If a random sample of 1000 chairs is inspected, what is the probability of there being no more than 3 defective?