UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Eng. M.Eng.

Mathematics E001: Mathematics

COURSE CODE : **MATHE001**

UNIT VALUE : **0.50**

DATE : **02-MAY-03**

TIME : **14.30**

TIME ALLOWED : **3 Hours**

03-C0935-3-160

TURN OVER

All questions may be attempted but only marks obtained on the best **seven** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. (a) Find the modulus and argument of $z = \frac{1}{2}(1 - i)$.
   Find z^8, expressing your answer in $x + iy$ form.
   Find $\sum_{n=0}^{\infty} z^n$, expressing your answer in $x + iy$ form.

   (b) Let z be a complex number with modulus 1 and argument θ. Show that

$$z^n + z^{-n} = 2\cos n\theta.$$

   Show that

$$\cos^5\theta = \frac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right).$$

   Hence find

$$\int_0^{\pi/3} \cos^5\theta d\theta.$$

2. (a) Define $\sinh x$ and $\cosh x$ in terms of exponentials, and sketch their curves. Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

   (b) Show that

$$\sinh^{-1} x = \ln\left[x + \sqrt{x^2 + 1}\right].$$

   Hence, or otherwise, find

$$\frac{d}{dx}(\sinh^{-1} x).$$

   Find

$$\int \sinh^{-1} x dx.$$

3. (a) Differentiate

$$\text{(i) } x^{\cos x} \qquad \text{and} \qquad \text{(ii) } x^2 \cos(e^{-x}).$$

(b) Let $x^3 - 4x + y^2 + y = 2$. Find the equation of the tangent at $x = 2$ which touches the curve at a positive value of y.

(c) Let $y = \sec t$ and $x = \tan t$. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ by parametric diffentiation. Using a trigonometric identity, find a relation between y and x. By differentiating this relation, check your answer to $\dfrac{dy}{dx}$ found above.
For $-\pi/2 < t < \pi/2$, where does the curve $y = \sec t$ and $x = \tan t$ have a turning point? Is it a maximum, minimum or point of inflection? Justify your answer.

4. Find the following integrals

(i) $\displaystyle\int \frac{x^2}{x^2-1} dx,$

(ii) $\displaystyle\int_0^2 \frac{1}{\sqrt{x^2+4x}} dx,$

(iii) $\displaystyle\int \frac{1}{x \ln x} dx,$

(iv) $\displaystyle\int e^{-x} \cos x dx.$

5. (a) Define carefully the dot and cross products of two vectors $\mathbf{a}$ and $\mathbf{b}$.

(b) The vectors $\mathbf{a} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{j} - 3\mathbf{k}$ form two sides of a triangle. Find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. Find the area of the triangle and the angle between $\mathbf{a}$ and $\mathbf{b}$.

(c) Given two non-zero vectors $\mathbf{c}$ and $\mathbf{d}$, show, using the dot product, that if $|\mathbf{c} + \mathbf{d}| = |\mathbf{c} - \mathbf{d}|$ then $\mathbf{c}$ and $\mathbf{d}$ are perpendicular.

6. (a) Let $f(x,y) = x^2 \exp\left(\frac{x}{y}\right)$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(b) Let $f(x,y) = xy\cos(xy)$. Verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

(c) Let $z = x^3 f\left(\frac{x}{y}\right)$ where f is any differentiable function. Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z.$$

7. Solve the following differential equations:

(a) $e^{y^2} x\frac{dy}{dx} = y^{-1}, \quad y(1) = 2;$

(b) $\frac{dy}{dx} - 2y = x^4 e^{2x}, \quad y(0) = 1.$

8. (a) Find the general solution of the differential equation

$$y'' + 3y' + 5y = x;$$

(b) Solve the following initial-value problem

$$y'' - 3y' + 2y = e^{3x}, \quad y(0) = 0, \quad y'(0) = 1.$$

9. (a) Find the following limits:

(i) $\lim_{x\to 0} \frac{e^{2x} - 1}{3x}$;

(ii) $\lim_{n\to\infty} \frac{(n^5 + 2^n)^2}{2n^4 + 4^{n+1}}$.

(b) Determine whether the following series are convergent or divergent, justifying your answer.

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$;

(ii) $\sum_{n=0}^{\infty} \frac{2^n}{1+3^n}$.

10. (a) Use the trapezium rule to find an approximate value of the integral

$$\int_1^2 e^x dx$$

by dividing the range of integration into five equal intervals. Compare your result with the exact answer.

(b) Write down the binomial distribution for x successes in n independent trials each with probability p of success.

(c) A tool hire shop has six lawn mowers which it hires out on a daily basis. The number of lawn mowers requested per day follows a Poisson distribution with mean 4.5. Find the probability that

(i) exactly three lawn mowers are hired out on any one day;

(ii) all lawn mowers are in use on any one day.