UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Eng. *M.Eng.*

Mathematics E001: Mathematics

COURSE CODE	:	**MATHE001**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-02**
TIME	:	**10.00**
TIME ALLOWED	:	**3 hours**

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TURN OVER

All *questions may be attempted but only marks obtained on the best* **seven** *solutions will count. The use of an electronic calculator* **is** *permitted in this examination.*

1. (a) Find all solutions, in the form $x + iy$, of

$$z^4 + 2z^2 + 4 = 0.$$

Plot your solutions in the complex plane.

(b) Use complex numbers to find

$$\int e^{-x} \sin 2x dx.$$

2. (a) Define $\sinh x$ and $\cosh x$ in terms of exponentials. Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

(b) Show that

$$\sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right].$$

Letting $y = \sinh^{-1} x$, differentiate the above relation to show

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}}.$$

(c) Use the fact that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}},$$

to check you answer obtained in 2(b).

3. (a) Differentiate the following with respect to x:

(i) $\ln\left[x^2 \sin\left(e^x\right)\right]$,

(ii) $2^{\tan x}$.

(b) Let $y = x^2 e^x$. Using Leibniz's rule, find $\dfrac{d^n y}{dx^n}$.

(c) Let $x = \cosh 2t$ and $y = \cosh t$. Show that

$$\frac{dy}{dx} = \frac{1}{4y}.$$

Hence, or otherwise, find $\dfrac{d^2 y}{dx^2}$.

4. (a) Find the following integrals:

(i) $\displaystyle\int \frac{x^2+1}{x^2-1} dx$,

(ii) $\displaystyle\int \frac{dx}{x\left(\ln x\right)}$,

(iii) $\displaystyle\int_0^2 \frac{dx}{\sqrt{3+2x-x^2}}$.

(b) Let

$$\mathrm{I}_{2n} = \int_0^{\pi/2} \sin^{2n} x \, dx,$$

where n is a positive integer. Show that

$$\mathrm{I}_{2n} = \frac{2n-1}{2n} \mathrm{I}_{2n-2} \, .$$

[Hint : write $\sin^{2n} x = \sin^{2n-1} x \sin x$, and use integration by parts.]

Deduce

$$\mathrm{I}_{2n} = \frac{(2n-1)(2n-3)\ldots 1}{(2n)(2n-2)\ldots} \frac{\pi}{2}.$$

Hence find

$$\int_0^{\pi/2} \sin^8 x dx.$$

5. (a) Define the dot product $\mathbf{a} \cdot \mathbf{b}$ and the cross product $\mathbf{a} \times \mathbf{b}$ of vectors $\mathbf{a}$ and $\mathbf{b}$.

Show that

$$|\mathbf{a}|^2 |\mathbf{b}|^2 = (\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}),$$

where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the magnitudes of vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

(b) Use the cross product to find the area of the triangle with sides $\mathbf{a} = -\mathbf{i} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$. Use the dot product to find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

(c) If $\mathbf{c} - \mathbf{d}$ and $\mathbf{c} + \mathbf{d}$ are perpendicular, prove that $|\mathbf{c}| = |\mathbf{d}|$.

6. (a) Let $f(x, y) = (x - y)\cos\left(\frac{y}{x}\right)$. Find

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y}.$$

(b) Show that $f(x, t) = \exp(-\beta t)\sin(\alpha x)$ is a solution of the equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

when α, β are constants such that $\beta - \alpha^2 = 0$.

(c) Let $z = y^2 f\left(\frac{x}{y}\right)$ where f is any differentiable function. Show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

7. Solve the following differential equations:

(a) $(x + 1)\sin y \frac{dy}{dx} = 1, \quad y(0) = 0;$

(b) $\frac{dy}{dx} + \frac{2}{x}y = 1, \quad y(1) = 1.$

8. (a) Find the general solution of the differential equation

$$y'' + 3y' + 2y = \cos x.$$

(b) Solve the following initial-value problem

$$y'' - y' + y = x + 1, \quad y(0) = y'(0) = 0.$$

9. (a) Find the following limits:

(i) $\lim_{x \to 0} \frac{\sin x - x}{3x^3}$;

(ii) $\lim_{n \to \infty} \frac{n^3 + 9^n}{(n^3 + 3^n)^2}$.

(b) Determine whether the following series are convergent or divergent, justifying your answer.

(i) $\sum_{n=0}^{\infty} \frac{1+n}{1+n^2}$;

(ii) $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$..

10. (a) Use Newton's method to find an approximate solution of the equation,

$$x = \frac{1}{1+x^2},$$

to three decimal places, starting from $x = 1$.

(b) Write down the Poisson probability distribution with mean μ.

(c) On average, 0.5% of all manufactured nails are defective. If a random box of 500 nails is inspected, what is the probability of there being no more than 3 defective nails?

END OF PAPER