UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics C399: Mathematics in Biology I

COURSE CODE : **MATHC399**

UNIT VALUE : **0.50**

DATE : **23-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1029-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. In a population model, the population density is N, and the per capita birth and death rates are $\beta(N)$ and $\delta(N)$ respectively where

$$\beta(N) = \frac{rN}{K + N^2}, \quad \delta(N) = d, \quad \text{where } r, d, K > 0 \text{ are constants.}$$

   (a) Write down the differential equation for the population growth.

   (b) Show that there is a unique locally asymptotically steady state population when $\mu = \dfrac{r}{d\sqrt{K}} < \mu^*$ for some μ^* which you should find. Find and classify the stability of all the steady states of the model when $\mu > \mu^*$.

   (c) Suppose that initially $\mu = 3\mu^*/2$ and the population is steady and at its maximum stable size. The population then experiences a severe and sustained famine during which the per capita death rate doubles. After a long period the famine lifts and food resources are restored. Explain what happens to the population during this sequence of events.

2. In an age-structured population, the number of females of age a at time t is $N_a(t)$ for $a, t = 0, 1, 2, \ldots$. The maximum age any individual can reach is 6. The probability of surviving from age a to age $a+1$ is p_1 for $0 \leqslant a \leqslant 2$ and p_2 for $3 \leqslant a \leqslant 5$ where $p_1, p_2 \in (0, 1)$ are constants. Females reach sexual maturity at the age $a = 5$ beyond which the expected number of offspring to an individual is a constant $b > 0$.

   (a) Derive the Euler-Lotka equation for the stable population growth rates λ and show using a graph that the equation has a unique positive root λ_0. Are the other roots real or complex?

   (b) When $p_1 = p_2 = p$ find an explicit form for the stable age structure in terms of λ_0 and p.

3. A population is modelled by the differential equation

$$\frac{dN}{dt} = \rho N\left(1 - \frac{N}{K(t)}\right), \quad N(0) = N_0, \tag{1}$$

where $\rho > 0$ and $N_0 \geqslant 0$ are constants and $K(t)$ is a periodic function with period T and satisfying $K(t) \geqslant \bar{K}$ for all $t \geqslant 0$, for some $\bar{K} > 0$.

(a) Give an ecological interpretation of ρ and $K(t)$.

(b) By considering $M(t) = N(t)\, e^{-\rho t}$, or otherwise, show that (1) has the solution

$$N(t) = \frac{N_0 e^{\rho t}}{1 + N_0 \int_0^t H(u)\, du}, \qquad \text{where } H(u) = \frac{\rho}{K(u)} e^{\rho u}.$$

(c) Show that the asymptotic population behaviour N_∞ defined by $N_\infty(s) = \lim_{k\to\infty} N(kT + s)$ is periodic with period T and find the value of

$$\int_0^T \frac{N_\infty(s)}{K(s)}\, ds.$$

4. In a model of predators density P feeding on a prey density N the dynamics takes the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - N\phi(N,P), \quad \frac{dP}{dt} = P(\sigma(N) - \mu), \tag{2}$$

where $r, \mu, K > 0$ are constants and, for some $\epsilon > 0$, $\sigma'(N) \geqslant \epsilon$ for $N \geqslant 0$ and $\frac{\partial \phi}{\partial P}(N,P) \geqslant \epsilon$ for $N, P \geqslant 0$.

(a) Explain briefly the ecological meaning of ϕ and σ. What additional conditions should σ and ϕ satisfy to render the model realistic?

(b) Derive conditions under which (2) has a unique positive steady state (N^*, P^*).

(c) Find the condition for (N^*, P^*) to be locally asymptotically stable.

(d) For the case when $\sigma(N) = \bar{\sigma}N$ for $\bar{\sigma} > 0$ constant and

$$\phi(N,P) = \frac{\gamma P}{N^2 + \alpha}, \quad \alpha, \gamma > 0 \text{ constants},$$

show that if $K^2 > 3\alpha$ a stable limit cycle appears when μ increases through $\bar{\mu}$ for some $\bar{\mu}$ which you should find. (You may assume that (N^*, P^*) is asymptotically stable when $\mu = \bar{\mu}$).

5. A population has density N_t at t years and the population at $t+1$ years is given by

$$N_{t+1} = F(N_t). \tag{3}$$

(a) Explain what is meant by local asymptotic stability of a steady state of (3). What is the condition that guarantees such a steady state is locally asymptotically stable?

(b) Interpret all solutions N of $F(F(N)) = N$.

Now let $F(N) = \dfrac{rN}{A + N^m}$ where $r, A > 0$ are real constants and $m > 0$ an integer.

(c) Find and classify all the steady states of (3).

(d) When $m = 3$ and $A = 1$ show that a 2-cycle appears in (3) as r exceeds 3. (*Hint: You may find it helpful to consider* $x = 1 + N^3$)