All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Briefly explain why non-overlapping generations can lead to a discrete time model for the growth of a population.

   (i) An insect population grows according to the discrete time model

$$N_{t+1} = \frac{rN_t^2}{1+N_t^2}.$$

   Find all the steady state populations and analyse their linear stability.

   (ii) Sketch the cobweb map for N_t when $r > 2$.

   (b) In a continuous time population model with population density $N(t)$ at time t the net *per capita* reproductive rate $\beta(N)$ is

$$\beta(N) = \frac{rN}{K+N^3},$$

   and the *per capita* death rate is a constant $d > 0$.

   (i) Write down the differential equation governing the population dynamics and show that for $K < 2\left(\frac{r}{3d}\right)^{3/2}$ there are three possible steady states. (Hint: sketch the curve $F(N) = N^3 - (r/d)N + K$).

   (ii) By sketching dN/dt as a function of N, or otherwise, determine the stability of these steady states.

2. (a) A model for the interaction of 2 species is given by

$$\frac{1}{N_1}\frac{dN_1}{dt} = \rho_1\left(1 - \frac{N_1}{K_1}\right) - c_1 N_2 \qquad (1)$$

$$\frac{1}{N_2}\frac{dN_2}{dt} = \rho_2\left(1 - \frac{N_2}{K_2}\right) - c_2 N_1 \qquad (2)$$

where $\rho_1, \rho_2, K_1, K_2, c_1, c_2 > 0$. What do the parameters K_1, K_2, c_1, c_2 represent in ecological terms?

(b) Show that with $\tau = \rho_1 t$, $u_1 = N_1/K_1$ and $u_2 = N_2/K_2$ the equations (1), (2) can be rewritten in the form

$$\frac{1}{u_1}\frac{du_1}{d\tau} = 1 - u_1 - \alpha_{12}u_2 \qquad (3)$$

$$\frac{1}{u_2}\frac{du_2}{d\tau} = r(1 - u_2 - \alpha_{21}u_1) \qquad (4)$$

where $r, \alpha_{12}, \alpha_{21}$ are constants which you should find.

(c) Find all the steady states of (3), (4), stating clearly the conditions for a non-zero steady state to exist.

(d) Determine the stability of the steady states when the parameters α_{12} and α_{21} are such that a non-zero steady state exists.

(e) Sketch the $(u_1, u_2)-$phase plane for the case $\alpha_{12} < 1$, $\alpha_{21} < 1$.

(f) Now suppose that $c_1 < 0$ and $c_2 < 0$. What does this model ecologically? Determine conditions on α_{12} and α_{21} for there to be a stable coexistence of the two species.

(g) What are the possible final states of the populations u_1 and u_2 when $c_1 < 0$ and $c_2 < 0$? Sketch the $(u_1, u_2)-$phase plane to illustrate each outcome. Is the model realistic?

3. In a predator-prey model the interaction of the prey N with the predator P is given by

$$\begin{aligned} \frac{dN}{dt} &= N\rho(N) - \phi(N,P) \quad (5) \\ \frac{dP}{dt} &= \alpha\phi(N,P) - \mu P \quad (6) \end{aligned}$$

where $\mu, \alpha > 0$.

(a) Explain the biological meaning of the terms in (5), (6). What are the two conditions on $\rho(N)$ that ensure that in the absence of predation there is a locally asymptotically stable prey population K?

Now suppose $\rho(N) = r(1 - N/K)$ for $\sigma > 0$ and $\phi(N,P) = \dfrac{\gamma NP}{A+N}$ for $\gamma, A > 0$.

(b) Show that if $0 < \dfrac{\mu A}{\alpha\gamma - \mu} < K$ there are 3 steady states and find them.

(c) Show that when it exists the non-zero steady state is linearly stable when

$$K < A\left(\frac{\alpha\gamma + \mu}{\alpha\gamma - \mu}\right),$$

and determine the linear stability of the other two steady states.

(d) Sketch the phase plane when

$$0 < \frac{\mu A}{\alpha\gamma - \mu} < K < A\left(\frac{\alpha\gamma + \mu}{\alpha\gamma - \mu}\right).$$

4. In an age-structured population there are n age classes, and the density at age k at time t is denoted by $N_k(t)$ and $\mathbf{N}(t) = (N_1(t), \ldots, N_n(t))^T$. The expected number of offspring of a female at age k is b_k and the probability that an individual aged $k \geqslant 0$ (with $k = 0$ the newborns) survives to age $(k+1)$ is p_k.

   (a) Show that

$$\mathbf{N}(t+1) = L\mathbf{N}(t) \; t = 1, 2, \ldots \text{ where } L = \begin{pmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & p_{n-1} & 0 \end{pmatrix}$$

   is the Leslie matrix for the population model and $f_k = p_0 b_k$ for $k \geqslant 1$.

   (b) Write down the corresponding Euler-Lotka equation for the eigenvalues of L.

   (c) If Leslie matrix is aperiodic and has distinct eigenvalues what can you say about the asymptotic growth rate of the age classes? If the asymptotic age structure of the population is stationary what can you say about the value of the mean fitness λ_0?

   In an animal population of juveniles J and adults A, juveniles mature into sexually mature adults at age m. The probability ℓ_s of juvenile surviving to some threshold age $s \leqslant m$ is small, but beyond this age the probability p_k of surviving from age k to age $k+1$ is a constant p, i.e. $p_k = p$ for $k \geqslant s$. The expected fecundity $b_k = \beta_0(1 - e^{-\alpha m})$ for all $k \geqslant m$ and $b_k = 0$ for $k < m$.

   (d) Show that the characteristic polynomial for the eigenvalues of the Leslie matrix (i.e. the Euler-Lotka equation) for this model is

$$\lambda^m - p\lambda^{m-1} - \mu\beta_0(1 - e^{-\alpha m})p^m = 0 \text{ where } \mu = \ell_s p^{-s}.$$

   (e) What is meant by an optimal life history strategy with respect to m?

   (f) Let m^* be the optimal life history strategy and suppose that the stable age structure reached for $m = m^*$ is stationary. Show that

$$m^* = \frac{1}{\alpha} \log\left(1 - \frac{\alpha}{\log p}\right).$$

5. In a predator-prey model, the predator P and prey N interact according to

$$\begin{aligned}\frac{1}{N}\frac{dN}{dt} &= a - bP - eN \\ \frac{1}{P}\frac{dP}{dt} &= cN - fP - d\end{aligned}$$

where $a, b, c, d, e, f > 0$ are all constants.

(a) Briefly explain the biological interpretation of the equations.

(b) Find all the steady states, showing that a non-zero steady state (N^*, P^*) is possible only if $ac > ed$.

(c) Classify the linear stability of the steady states when $ac > ed$.

(d) By considering the Lyapunov function

$$V(N,P) = \gamma_1 \left\{ N - N^* - N^* \log\left(\frac{N}{N^*}\right) \right\} + \gamma_2 \left\{ P - P^* - P^* \log\left(\frac{P}{P^*}\right) \right\},$$

for appropriately chosen γ_1, γ_2, show that the non-zero steady state (N^*, P^*) is globally asymptotically stable.

(e) What happens when both $e = 0$ and $f = 0$? Sketch the phase plane in this case.