UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics C399: Mathematics in Biology I

COURSE CODE : **MATHC399**

UNIT VALUE : **0.50**

DATE : **13–MAY–04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04–C0941–3–30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Suppose that birth rate at time $t \geqslant 0$ in a population is $b(t) \geqslant 0$. Show that the probability $P(t)$ that no individual gives birth in $[0, t)$ is given by

$$P(t) = \exp\left(-\int_0^t b(s)\,ds\right).$$

Show that provided $tP(t) \to 0$ as $t \to \infty$ the expected time to the first birth is

$$\bar{T} = \int_0^\infty P(t)\,dt.$$

Now suppose that the birth rate is $b(t) = te^{-\lambda t^2}$.

(a) What is the probability that an individual will give birth sometime in their lifetime?

(b) What is the probability that an individual gives birth sometime given that they do not give birth in the time interval $[0, t)$?

[You may ignore deaths].

2. (a) Find the explicit solution of the Logistic equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \quad (r, K > 0, \text{ constants})$$

(b) By introducing the new rescaled time variable $\tau(t) = \int_0^t p(s)\,ds$, or otherwise, show that if $p(t)$ is periodic of period T then the solution to the time-dependent equation

$$\frac{dN}{dt} = p(t)N\left(1 - \frac{N}{K}\right), \quad (K > 0, \text{a constant})$$

is for $t = nT + s$ with $s \in [0, T)$,

$$N(nT + s) = \frac{N_0}{\frac{N_0}{K} + e^{-nRT}\left[1 - \frac{N_0}{K}\right] e^{-f(s)}} \qquad (0 \leqslant s < T),$$

where $R = \frac{1}{T}\int_0^T p(t)\,dt$ is the mean seasonal reproductive-rate, and $f(s) = \int_0^s p(t)\,dt$. How does the population evolve when $R = 0$?

(c) Suppose now that the instantaneous birth rate $b(t)$ and death rate $d(t)$ are given by

$$\text{for } k = 0, 1, \ldots. \begin{cases} b(t) = \begin{cases} 0 & (kT \leqslant t < (k+\beta)T) \\ b_0(T - t) & ((k+\beta)T \leqslant t < (k+1)T) \end{cases} \\ d(t) = d_0 e^{\alpha t} \quad (kT \leqslant t < (k+1)T), \end{cases}$$

Let $\mu = \dfrac{2d_0}{\alpha b_0(1-\beta)^2}$. Discuss the long term behaviour of the population for different values of μ and show that there is a critical value, which you should find, for which the population cycles.

3. In a simple model for a population of reptiles, the population is divided into 2 classes: the juvenile class J and the adult class A. Juveniles cannot reproduce, and adults reproduce at the rate b_A. The probability of a juvenile surviving to adulthood is p_J and the probability of an adult surviving long enough to give birth is p_B.

(a) Derive equations for the size of the juvenile population J_{k+1} and adult population A_{k+1} at generation $k+1$ in terms of generation k ($k \geqslant 1$), and find the Leslie matrix for the model.

(b) If X_k is the juvenile fraction of the population, show that

$$X_{k+1} = \frac{(1 - X_k)}{\alpha X_k + (1 - X_k)} = F(X_k), \tag{1}$$

where α is a constant which you should find in terms of p_A, p_J and b_A.

(c) Find the unique steady state X^* of (1) and find its linear stability eigenvalue.

(d) Find all period 2 cycles of (1).

(e) Sketch the cobweb plot for (1) on $[0, 1]$.

4. A predator-prey model has the form

$$\frac{dU}{dt} = \rho U\left(1-\frac{U}{K}\right) - \phi(U,V)$$
$$\frac{dV}{dt} = V(\sigma U - \mu),$$

where $\phi(U,V) = \dfrac{\gamma UV}{A+U}$ and $\rho, K, \gamma, A, \alpha$ are all positive constants.

(a) Which of U, V represents the predator, and which the prey?

(b) Sketch $\phi(U,V)$ for fixed $V > 0$, and briefly explain the ecology that ϕ models.

(c) Show that the model can be written as

$$\frac{dx}{d\tau} = rx(1-x) - \frac{yx}{a+x}$$
$$\frac{dy}{d\tau} = y(x-\delta)$$

where x, y, τ are dimensionless variables and r, a, δ are dimensionless constants.

(d) Find the location and stability matrix for the non-zero steady state and show that a necessary condition for a limit cycle to appear around this steady state is that δ passes through a critical value which you should find.

(e) What would happen to the cycling populations if the carrying capacity of the prey then increased? Sketch the phase plane to justify your answer.

5. A disease agent invades a large population. Initially all members of the population are susceptible to the disease, and the disease is fatal. The birth rate for the population is $b > 0$ and in the absence of the disease the normal death rate is $d > 0$. The death rate due to the infection is $\delta > 0$.

(a) Denoting the susceptible population density by S and the infective population density by I, and assuming that the rate of infection of susceptibles is λSI, obtain equations for the population dynamics of I and S.

(b) Hence show that if $N = I + S$,

$$\begin{aligned} \frac{dN}{dt} &= (b-d)N - \delta I \\ \frac{dI}{dt} &= -(d+\delta)I + \lambda(N-I)I. \end{aligned}$$

(c) Show that the entire population dies out if $b < d$.

(d) Show that if $b > d$ there is a finite nonzero steady state (N^*, I^*) which is asymptotically stable if and only if $b - d < \delta$. What is the biological meaning of this condition?

(e) What happens if $b - d > \delta$. Sketch the phase plane in this case to justify your answer?