UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C399: Mathematics in Biology I

COURSE CODE	:	**MATHC399**
UNIT VALUE	:	**0.50**
DATE	:	**27-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0934-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Briefly state under what assumptions a discrete time model for a population is valid, and give an example.

   A model for the growth of a single species population of size N_t at time t is given by

$$N_{t+1} = F(N_t) := \frac{rN_t}{1+N_t^3}, \quad (r>0), \quad t = 0, 1, 2, \ldots. \tag{1}$$

   (b) Find the steady state population(s), and determine their local stability.

   (c) Sketch the cobweb maps for equation (1) for the cases (i) $r < 1$, (ii) $0 < r < 3$ and (iii) $r > 3$.

   (d) By introducing $x_t = 1 + N_t^3$, or otherwise, show that a 2-cycle orbit of equation (1) exists when all solutions of

$$q(x) = x^3 - r^2x^2 + r^3x - r^3 = 0$$

   are greater than one. Hence show that a non-trivial 2-cycle exists when r just exceeds 3.

2. Consider the population model for 2 species:

$$\begin{aligned} \frac{dN_1}{dt} &= N_1\left(1 + a_{11}N_1 + a_{12}N_2\right) \\ \frac{dN_2}{dt} &= N_2\left(1 + a_{21}N_1 + a_{22}N_2\right) \end{aligned} \tag{2}$$

where $a_{12} > 0$ and $a_{21} > 0$, but $a_{11} < 0$ and $a_{22} < 0$.

(a) What are the per capita net reproduction rates for each species?

(b) In ecological terms, what do each of the coefficients a_{ij} measure?

(c) Find conditions on the a_{ij} for there to be a steady state population (N_1^*, N_2^*) with $N_1^* > 0$, $N_2^* > 0$.

(d) By considering the function

$$V(N_1, N_2) = \left\{N_1 - N_1^* \log\left(\frac{N_1}{N_1^*}\right)\right\} + \left\{N_2 - N_2^* \log\left(\frac{N_2}{N_2^*}\right)\right\},$$

show that the non-zero steady state (N_1^*, N_2^*) is globally asymptotically stable.

(e) Sketch the phase plane, explaining the ecological significance of the non-zero steady state.

(f) Now suppose that for the system of equations (2) you are no longer told the sign of each of the a_{ij} for $i, j = 1, 2$: they could be positive, negative or zero. You are told, however, that solutions $(N_1(t), N_2(t))$ of the system (2) satisfy $A < N_1(t), N_2(t) < B$ for some constants $A, B > 0$ and $t > 0$, and that the system (2) has a unique positive steady state $(\bar{N}_1, \bar{N}_2)$ (i.e $\bar{N}_1 > 0$ and $\bar{N}_2 > 0$). Show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T N_i(t)\, dt = \bar{N}_i, \quad i = 1, 2.$$

(You may assume without proof that the limit in each integral exists.)

3. A species of insect has a life cycle in which larvae progress to pupae and finally onto adults. The larval and pupal periods are each 2 weeks. Both the larvae and pupae are cannibalistic, consuming eggs. Let $l(t), p(t)$ and $a(t)$ denote the larval, pupal and adult population densities at time t. The populations $\mathbf{x} = (l, p, a)^T$ are modelled according to the Leslie matrix system:

$$\mathbf{x}(t+1) = L\mathbf{x}(t), \quad t = 0, 1, 2, \ldots$$

(a) Obtain the Leslie matrix L in terms of the larval recruitment rate per unit time b, and the death rates μ_l (larval), μ_p (pupal), μ_a (adult). How long is the basic time unit in this model?

(b) Show that the characteristic polynonial for the eigenvalues of L is

$$c(\lambda) = -\lambda^3 + (1-\mu_a)\lambda^2 + b(1-\mu_l)(1-\mu_p),$$

and that $c(\lambda) = 0$ has only one real solution which is positive.

(c) Show that the population grows indefinitely if

$$b(1-\mu_l)(1-\mu_p) > \mu_a.$$

(d) When $b = 16$, $\mu_a = \mu_l = \frac{1}{2}$ and $\mu_p = \frac{1}{4}$ find the asymptotic age distribution.

4. Let $b(t)\delta t$ be the probability that the average individual in a given population gives birth in the infinitesimal time interval $[t, t+\delta t)$.

   (a) Given that the probability that an individual chosen at random from the population has at least one descendant at time t is

   $$1 - \exp\left(-\int_0^t b(s)\, ds\right),$$

   find the expected time, $\bar{T}$, for an individual to produce their first offspring.

   In the following, the birth rate $b(t)$ of a population of insects is given by

   $$b(t) = \begin{cases} b & \text{if } \lfloor t \rfloor \leqslant t \leqslant \lfloor t \rfloor + \alpha \\ 0 & \text{if } \lfloor t \rfloor + \alpha < t \leqslant \lfloor t \rfloor + 1 \end{cases}$$

   where $\lfloor t \rfloor$ denotes the largest integer that does not exceed t, and $b > 0$, $1 \geqslant \alpha \geqslant 0$ are constants.

   (b) Show that the expected time $\bar{T}$ for an individual to produce their first offspring is

   $$\bar{T} = \frac{1}{b} + \frac{(1-\alpha)e^{-b\alpha}}{1 - e^{-b\alpha}}.$$

   (c) How does $\bar{T}$ change if the birth rate function $b(t)$ is replaced by the periodic 'saw-tooth' function

   $$b(t) = b\,[1 - (t \bmod 1)], \quad t \geqslant 0.$$

5. A disease spreads within a closed community via infected individuals who infect susceptible individuals that have not previously had contact with the disease. The infected individuals either recover and then acquire immunity from the disease, or they die. A simple model for such a process is the set of differential equations:

$$\frac{dS}{dt} = -rSI \tag{3}$$
$$\frac{dI}{dt} = rSI - aI \tag{4}$$
$$\frac{dR}{dt} = aI. \tag{5}$$

(a) Give an interpretation of the model, stating what the variables S, I, R and parameters $r > 0$ and $a > 0$ represent. What assumptions does the model make about the birth and death rates?

(b) Show that the total population over all the classes remains constant.

(c) If initially there are a small number $I_0 > 0$ of infectives, what is the condition for an epidemic to begin?

(d) Show that

$$S(t) + I(t) - \frac{a}{r} \log S(t) = \text{constant}$$

along a solution $(S(t), I(t), R(t))$.

(e) Sketch the $S - I$ phase plane for equations (3) and (4).

The model is modified to include an extra term in equations (3) and (5) to give

$$\frac{dS}{dt} = -rSI - v_0 S \tag{6}$$
$$\frac{dI}{dt} = rSI - aI \tag{7}$$
$$\frac{dR}{dt} = aI + v_0 S. \tag{8}$$

where $v_0 > 0$ is a constant.

(f) What disease control strategy might these extra terms model?

(g) Prove that if $S(0) \geqslant 0$, $I(0) \geqslant 0$ then $S(t) \geqslant 0, I(t) \geqslant 0$ for $t \geqslant 0$ and that in the long term all individuals either gain immunity to the disease or they die.

(h) Sketch the $S - I$ phase plane for case $v_0 > 0$ for equations (6) and (7) and compare it with the case $v_0 = 0$, commenting on the effect of the disease control strategy in epidemiological terms.