UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.A.   *B.Sc.*   *B.Sc.(Econ)*

Mathematics B611: Mathematics for Students of Economics, Statistics & Related Disciplines

COURSE CODE : **MATHB611**

UNIT VALUE : **0.50**

DATE : **11-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1003-3-150

TURN OVER

Answer ALL questions from Section A.
All questions from Section B may be attempted but only marks obtained on best two solutions from Section B will count.
The use of an electronic calculator is **not** *permitted in this examination.*

Section A: Use a separate answer book for this section

1. (a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$. Explain what is meant by saying that

   (i) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent;

   (ii) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ span $\mathbb{R}^n$;

   (iii) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ form a basis of $\mathbb{R}^n$;

   (iv) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ form an orthogonal basis of $\mathbb{R}^n$.

   (b) Determine which of the following sets form a vector space. The set consisting of all vectors of the form:

   (i) $\begin{pmatrix} x \\ 3x \\ 4x+7y \end{pmatrix}$;

   (ii) $\begin{pmatrix} x \\ y \end{pmatrix}$ with $x < 0$, $y < 0$;

   (iii) $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ such that $x + y - z = 1$.

   In case the set is not a vector space explain why not.

   (c) Show that if V, W are two subspaces in $\mathbb{R}^n$ whose intersection is $\mathbf{0}$, then $\dim(V) + \dim(W) = \dim(V + W)$.

2. (a) For an $m \times n$ matrix A, $\mathbf{b} \in \mathbb{R}^m$ and the augmented matrix $A_{\mathbf{b}}$, which of the following statements are true and which are false?

   (i) If $A\mathbf{x} = \mathbf{b}$ has a unique solution then $n = m$.

   (ii) If $A\mathbf{x} = \mathbf{0}$ has a unique solution then $n = m$.

   (iii) If $A\mathbf{x} = \mathbf{b}$ has a unique solution then $\text{rank}(A) = \text{rank}(A_{\mathbf{b}})$.

   (iv) If $\text{rank}(A) = \text{rank}(A_{\mathbf{b}})$ then $A\mathbf{x} = \mathbf{b}$ has a unique solution.

   (v) If $A\mathbf{x} = \mathbf{b}$ has a unique solution then $\text{rank}(A) = n$.

(b) Define $\ker A$ and show that it is a vector space. What is its dimension? Prove your answer.

(c) State Cramer's rule for the solution of a linear system $A\mathbf{x} = \mathbf{b}$. When is this rule applicable?

3. (a) Define the rank of a matrix.

(b) Show that if AB is defined and A is not singular, then $\text{rank}(AB) = \text{rank}(B)$. (You may use without proof that multiplying a matrix with another matrix cannot increase its rank.)

(c) Find the rank of A, B, C, where $C = AB$ and

(i) $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix};$

(ii) $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix};$

(iii) $A = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 0 \\ -1 & 0 \end{pmatrix};$

(iv) $A = \begin{pmatrix} 4 & 0 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}.$

(d) Is it true that if $AB = AC$ and $B \neq C$ then A is singular? Prove your answer.

Section B: Use a separate answer book for this section

4. (a) Explain what an inversion is and what it means that a permutation is even or odd.
   (b) Show that the transposition of two elements in any permutation changes odd to even and even to odd.
   (c) Define the determinant of an $n \times n$ matrix.
   (d) If A is a 3×3 matrix with $\det A = 2$ and B is a 3×3 matrix with $\det B = 5$, respectively, and if A and B differ only in their second row, what is
      (i) the determinant of $10AA^T$;
      (ii) the determinant of $A + B$?

5. (a) Give the definition of an eigenvalue and an eigenvector of an $n \times n$ matrix A.
   (b) Define the characteristic polynomial and explain why the eigenvalues of A are the roots of the characteristic polynomial.
   (c) Suppose that the matrix A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Show that:
      (i) The eigenvalues of $2A$ are $2\lambda_1, 2\lambda_2, \ldots, 2\lambda_k$;
      (ii) The eigenvalues of A^2 are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2$.
   (d) Show that if B is invertible then the characteristic polynomial of $B^{-1}AB$ and A coincide.

6. (a) Define what it means that an $n \times n$ matrix Q is orthogonal.
   (b) Show that $(Q\mathbf{x}, Q\mathbf{y}) = (\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
   (c) Deduce that Q preserves angle and length.
   (d) Show that every eigenvalue of Q has absolute value 1.