UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)*

Mathematics B611: Mathematics for Students of Economics, Statistics & Related Disciplines

COURSE CODE : **MATHB611**

UNIT VALUE : **0.50**

DATE : **20-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0918-3-160

TURN OVER

Answer ALL questions from Section A.
All questions from Section B may be attempted, but only marks obtained on the best **two** *solutions from Section B will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Section A

1. (a) Explain what is meant by an elementary row operation and by a row echelon form. How can these be used to compute the rank of a matrix? How does the determinant of a square matrix change under elementary row operations? How can the determinant be computed using elementary row operations?

   (b) Find a row echelon form of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & -1 & 4 & 3 & 2 \\ -1 & -4 & 4 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & -2 & 5 \\ 3 & 8 & 2 & 8 & 8 & 8 \end{pmatrix}.$$

   What is the rank of A?

2. (a) Give the definition of an eigenvalue, and an eigenvector of an $n \times n$ matrix A with real or complex entries.

   (b) Explain why the eigenvalues of A are the roots of the characteristic polynomial of A.

   (c) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix}.$$

   (d) Show: If λ is an eigenvalue of an invertible $n \times n$ matrix A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

3. (a) For an $m \times n$ matrix A and $b \in \mathbb{R}^m$, which of the following statements are true and which are false?

   (i) If the linear system $Ax = b$ has a unique solution x then $n = m$.

   (ii) If the linear system $Ax = b$ has a solution x then the rank of A equals the rank of the augmented $m \times (n+1)$ matrix (Ab).

   (iii) If $m > n$ and $b \neq 0$ then $Ax = b$ has no solution.

   (iv) If $m = n$ and $\det A \neq 0$ then $Ax = b$ has a unique solution for every b.

   (v) If $m < n$ then $Ax = 0$ has a nontrivial solution.

(b) State Cramer's rule for the solution of a linear system $Ax = b$. When is this rule applicable?

(c) Using Cramer's rule, find y (only) from the system

$$\begin{aligned} x + y &= 1 \\ 3x - z &= -4 \\ -3y + z &= 3 \quad . \end{aligned}$$

Section B

4. (a) Let A be an $n \times n$ matrix. Give the definition of the inverse matrix A^{-1}.

   (b) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}.$$

   (c) Assume A and B are $n \times n$ invertible matrices. Prove that AB is also invertible and find its inverse.

5. Let Q be an $n \times n$ matrix with $Q^T Q = I$ (i.e., an orthogonal matrix).

   (a) Show that $Qx \cdot Qy = x \cdot y$ for all $x, y \in \mathbb{R}^n$.

   (b) Show that Q maps orthogonal vectors to orthogonal vectors, i.e., $x \perp y$ implies $Qx \perp Qy$.

   (c) Show that $|Qx| = |x|$ for all $x \in \mathbb{R}^n$.

   (d) Show that every eigenvalue of Q has modulus 1.

   (e) Find real numbers x, y such that the matrix $Q = \begin{pmatrix} x & -1 \\ 1 & y \end{pmatrix}$ is orthogonal. What is the geometric meaning of the linear transformation induced by this matrix Q?

6. (a) Define the following terms: subspace of $\mathbb{R}^n$, basis, dimension.

   (b) For an $m \times n$ matrix A the null space of A is the set of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$. Show that this is a subspace of $\mathbb{R}^n$.

   (c) Compute the dimension and find a basis of the null space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 3 & 1 \end{pmatrix}.$$