UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)*

Mathematics B611: Mathematics for Students of Economics, Statistics & Related Disciplines

COURSE CODE	:	**MATHB611**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0914-3-120

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Explain what is meant by an elementary row operation, by the echelon form and by the reduced row echelon form of a matrix.

   (b) Find the reduced row echelon form of the matrix

$$A = \begin{pmatrix} 5 & 8 & 5 & -2 & -3 & 1 \\ 1 & 2 & 3 & 4 & -1 & 0 \\ 4 & 6 & 2 & -6 & -2 & 1 \\ 0 & 0 & 0 & 1 & -4 & 5 \\ 6 & 10 & 8 & 3 & -8 & 6 \end{pmatrix}.$$

   What is the rank of A?

2. (a) Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in a vector space V. Explain what is meant by saying that

   (i) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent;

   (ii) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span the vector space V;

   (iii) $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ form a basis of V.

   (b) Let the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be linearly independent. Put $\mathbf{w}_1 = \mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{w}_3 = \lambda\mathbf{v}_3 - \mathbf{v}_1$, where λ is a real number. For which values of λ are the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ linearly independent?

   (c) Find a basis for the vector space V consisting of all vectors $\begin{pmatrix} x \\ y \\ z \\ s \\ t \end{pmatrix} \in \mathbb{R}^5$ satisfying $x - y = z + s = t$. What is the dimension of V?

3. (a) Give the definition of an elementary matrix. State which of the following matrices are elementary:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) Show that each elementary matrix is invertible and its inverse is an elementary matrix.

(c) Give an example of two elementary matrices whose product is an elementary matrix different from the identity.

4. (a) Let A be an $n \times n$ matrix. Give the definition of the inverse matrix A^{-1}. Prove that the inverse matrix, if it exists, is unique.

(b) Let A be an $n \times n$ matrix. Prove that the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$ if and only if A is invertible.

(c) Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & 0 & 0 \\ 2 & -2 & 4 \end{pmatrix}.$$

5. (a) Define the scalar product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and the length of a vector $\mathbf{x} \in \mathbb{R}^n$.

(b) State the Cauchy-Schwarz inequality. State and prove the triangle inequality.

(c) Assume $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $|\mathbf{x}| = 1$, $|\mathbf{y}| = 4$, and $(\mathbf{x}, \mathbf{y}) = -2$. What is the length of $\mathbf{x} - \mathbf{y}$? What is the angle between $\mathbf{x}$ and $4\mathbf{x} + \mathbf{y}$?

6. (a) Explain how the Gram-Schmidt orthogonalization procedure works.

(b) Compute the Gram-Schmidt orthogonalization of the vectors

$$\mathbf{a}_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(c) Assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthonormal basis of $\mathbb{R}^n$. Let A be the $n \times n$ matrix whose kth column is $\mathbf{v}_k$. Show that $A^T A$ is the identity matrix.

7. (a) Define the determinant of an $n \times n$ matrix.

(b) Show that an $n \times n$ matrix A is invertible if and only if its determinant is non-zero.

(c) Let B be the following matrix:

$$B = \begin{pmatrix} 1 & 1 & 1 & 2 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & -2 & 1 \\ 2 & 3 & -3 & 1 \end{pmatrix}.$$

Compute its determinant. Are the rows of B linearly independent or not?