UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B8: Mathematics For Physics And Astronomy

COURSE CODE	:	**MATHB008**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

05-C0911-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find the radius of convergence of the following series

$$\text{(i)} \quad \sum_{n=0}^{\infty}(-1)^n n^2 z^n, \qquad \text{(ii)} \quad \sum_{n=0}^{\infty}\frac{(n!)^3 z^n}{(3n)!}, \qquad \text{(iii)} \quad \sum_{n=0}^{\infty}\frac{(-1)^{n+1}z^n}{n^n}.$$

(b) State de Moivre's Theorem and use it to show that

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}.$$

(c) Show that

$$\tan^{-1} z = \frac{1}{2\mathrm{i}}\ln\frac{1+\mathrm{i}z}{1-\mathrm{i}z},$$

and use it to find all values of $\tan^{-1}(-3\mathrm{i})$.

2. (a) Explain what is meant by the statement "The function $f(z)$ is analytic in the region $\mathcal{D}$ of the complex plane".

(b) If $f(z)$ is analytic in the region $\mathcal{D}$ and $f(x+iy) = u(x,y)+iv(x,y)$, write down the Cauchy-Riemann equations satisfied by $u(x,y)$ and $v(x,y)$ in $\mathcal{D}$. Hence, or otherwise, prove that $u(x,y)$ and $v(x,y)$ are harmonic in $\mathcal{D}$.

(c) Show that the function $f(z) = \cos(z)$ is analytic everywhere in the complex plane.

(d) Find λ such that the function $h(x,y) = (2x^2 - \lambda y^2 - y)$ is harmonic for all x and y. Determine (to an arbitrary constant) the analytic function $g(z)$ whose imaginary part is $h(x,y)$.

3. (a) **Derive Cauchy's Integral Formula** that states for a function $f(z)$ analytic and single-valued everywhere on and inside the contour C, the integral

$$\oint_C \frac{f(z)}{(z-z_0)}\mathrm{d}z,$$

is equal to $2\pi \mathrm{i} f(z_0)$ if C encloses the point z_0.
(*You may assume Cauchy's Theorem for the proof*).

(b) Use Cauchy's integral formula for derivatives to show that

$$\frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{e}^{tz}}{z^{n+1}}\mathrm{d}z = \frac{t^n}{n!}$$

where C is the circle $|z| = 1$.

(c) What is the value of

$$\oint_C \frac{\cos z}{z^3 + 2\mathrm{i}z^2 - z}\mathrm{d}z,$$

if C is

(i) the circle $|z| = \frac{1}{2}$
(ii) the circle $|z + 2\mathrm{i}| = 3$

4. Use suitable contours in the complex plane and the residue theorem to evaluate

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 + 3\sin\theta} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^6}\mathrm{d}x.$$

5. (a) If Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

is satisfied by the extremal of the functional

$$I = \int_a^b F(y, y')\mathrm{d}x,$$

where F has *no explicit dependence on* x, the prime denotes differentiation with respect to x, and $y(a)$ and $y(b)$ are prescribed, show that

$$F - y'\frac{\partial F}{\partial y'}$$

is constant.

(b) Find the extremal curve of

$$I = \int_1^2 \left\{x^2(y')^2 + 2y^2\right\} \mathrm{d}x,$$

if $y(1) = 4$ and $y(2) = 1$. Show that $I = 28$ on this extremal.

6. A uniform heavy chain is suspended from two fixed points, $(-a, 0)$ and $(+a, 0)$ in the usual (x, y) plane, with gravity acting in the direction of negative y. The length of the chain is $2L > 2a$. Show that if the chain hangs in a curve with equation $y = y(x)$, then

$$\int_{-a}^{+a} \sqrt{1 + (y')^2}\mathrm{d}x = 2L.$$

If the heavy chain is of density ρkgm^{-1} and the acceleration due to gravity is $g$ms^{-2}, show also that the potential energy V of the chain is given (to within an arbitrary constant) by

$$V = \rho g \int_{-a}^{+a} y\sqrt{1 + (y')^2}\mathrm{d}x.$$

Given that V is a minimum when the chain is in a stable equilibrium, deduce that the chain shape $y(x)$ satisfies

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \left(\frac{y - \lambda}{C}\right)^2 - 1,$$

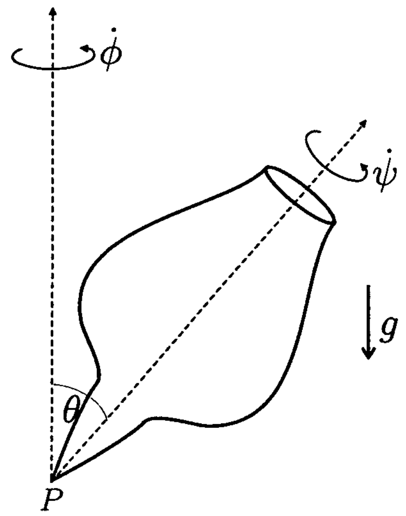
where λ and C are constants. Solve this equation to show that

$$y(x) = C\left\{\cosh(x/C) - \cosh(a/C)\right\},$$

is a solution satisfying the end conditions at $x = \pm a$. Finally, demonstrate that the unknown C satisfies the equation

$$\sinh(a/C) = L/C.$$

7. (a) Define the Lagrangian and Hamiltonian for a conservative system. Write down Lagrange's equations if the system has n degrees of freedom.



(b) A symmetric top is pivoted at a point P on its axis of symmetry at a distance a from its centre of mass. The generalised coordinates (Euler angles) θ, ϕ and ψ are defined as the angle between the vertical and the symmetry axis, the measure of rotation of the symmetry axis about the vertical, and the measure of the rotation of the top about its symmetry axis respectively (see figure).

You are given that the total kinetic energy T and potential energy V of the system are

$$T(\phi, \psi, \theta) = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2,$$
$$V(\phi, \psi, \theta) = mga\cos\theta,$$

where A and C are known constants (moments of inertia) and a dot indicates a derivative with respect to time.

Use Lagrange's Equations to derive the following conservation laws,

$$\dot{\psi} + \dot{\phi}\cos\theta = N,$$
$$A\dot{\phi}\sin^2\theta + CN\cos\theta = M,$$
$$A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + 2mga\cos\theta = 2E - CN^2,$$

where N, M and E are constants of the motion.

The top, initially spinning with angular velocity ω ($\dot{\psi} = \omega$) about its symmetry axis, is released from rest with its symmetry axis nearly vertical ($\theta \approx 0$ and $\dot{\phi} = 0$). The top is observed to move in such a way that the lowest position the symmetry axis reaches is the horizontal ($\theta = \pi/2$). Use the three conservation laws above to show that

$$\omega = \frac{\sqrt{2mgaA}}{C}.$$