UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B8: Mathematics For Physics And Astronomy

COURSE CODE : **MATHB008**

UNIT VALUE : **0.50**

DATE : **11-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0907-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) If $S_n = \sum_{k=0}^{k=n} az^k$, show that $S_n = \frac{a(1-z^{n+1})}{1-z}$ and use this result to show that

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos\left(\frac{n\theta}{2}\right)\sin\left(\frac{(n+1)}{2}\theta\right)}{\sin\left(\frac{\theta}{2}\right)}.$$

(b) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{z-i}{z+i}\right)^{2n}$$

converges only in the upper-half of the complex plane where $\mathrm{Im}(z) > 0$.

(c) Write down an expression for $\cos z$ in terms of $\mathrm{e}^{\mathrm{i}z}$ and use this to find all values of z such that $\cos z = 2$.

2. (a) If a function $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ is analytic in some region $\mathcal{D}$ of the complex plane, derive the Cauchy-Riemann equations satisfied by u and v in $\mathcal{D}$. Hence, or otherwise, prove that u and v are harmonic in $\mathcal{D}$.

(b) Show that the function $v(x, y) = \mathrm{e}^x(x\sin y + y\cos y)$ is harmonic for all x and y. Determine the analytic function $f(z)$, the imaginary part of which is $v(x, y)$ and where $f(0) = \mathrm{i}$.

3. (a) Write down Cauchy's Theorem and prove it using Green's Theorem, which states that for some simply connected region R enclosed by a curve C,

$$\oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\mathrm{d}x\mathrm{d}y,$$

for any two continuous functions $P(x, y)$ and $Q(x, y)$.

(b) Use Cauchy's Integral Formula with the unit circle $|z| = 1$ as the contour chosen in the complex plane to show that, if $f(z)$ is analytic within and on the unit circle, then

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta.$$

(c) What is the value of

$$\oint_C \frac{z - \mathrm{i}}{z^3 - z^4}\mathrm{d}z,$$

if C is

(i) the circle $|z| = \frac{1}{2}$?

(ii) the circle $|z + 1| = 3$?

4. Use the residue theorem and a suitable contour in the complex plane to show

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1 - a^2}}, \qquad \text{for real } a \text{ where } |a| < 1.$$

Using a different contour in the the complex plane, evaluate

$$\int_{-\infty}^{+\infty} \frac{\cos(x)}{(x^2 + b^2)(x^2 + c^2)}\mathrm{d}x,$$

where b and c are positive real numbers.

5. (a) If Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

is satisfied by the extremal of the functional

$$I = \int_a^b F(y, y')\mathrm{d}x,$$

where F has *no explicit dependence on* x, the prime denotes differentiation with respect to x, and $y(a)$ and $y(b)$ are prescribed, show that

$$F - y'\frac{\partial F}{\partial y'}$$

is constant.

(b) Consider the Brachistochrone problem of finding the curve $y(x)$ down which a particle will slide from the point (x_0, y_0) to the point (x_1, y_1), where $x_1 > x_0$ and $y_1 < y_0$, in the shortest possible time under gravity (which acts in the negative y-direction). **You are given** that the time T of descent is

$$T = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \frac{\sqrt{1 + (y')^2}}{\sqrt{y_0 - y}} \mathrm{d}x.$$

Show that if T is a minimum, the function $y(x)$ satisfies the differential equation

$$(y_0 - y)\left(1 + (y')^2\right) = D,$$

for some constant D. Use the substitution $y' = \cot\theta$ to show that

$$x = x_0 + \frac{D}{2}(\sin 2\theta - 2\theta),$$
$$y = y_0 - \frac{D}{2}(1 - \cos 2\theta),$$

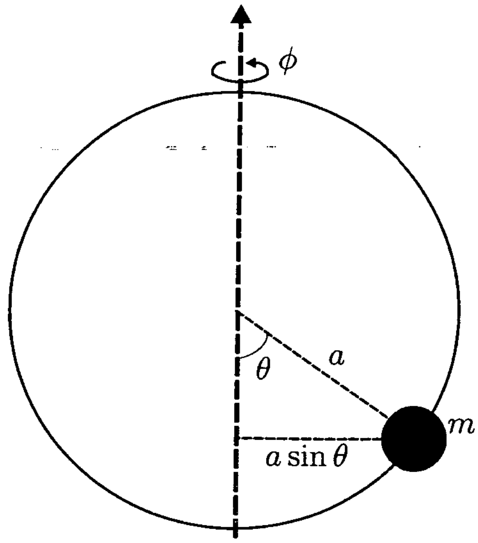
is a parametric form of the required curve. How can the constant D be determined?

6. Find the extremal curve for the integral

$$I = \int_0^{\pi/2} \left(y'^2 + 2xyy'\right) \mathrm{d}x,$$

subject to the constraints $y(0) = 0$, $y(\pi/2) = 1$ and $\int_0^{\pi/2} y\,\mathrm{d}x = \left(\frac{\pi}{2} - 1\right)$.

7. (a) Define the Lagrangian and Hamiltonian for a conservative system. Write down Lagrange's equations if the system has n degrees of freedom.



(b) A small bead of mass m slides freely on a smooth uniform circular wire of radius a. The wire is *free* to rotate about its vertical diameter and has moment of inertia of $Ma^2/2$ about the rotation axis. Gravity acts downwards in the vertical direction with acceleration g. The system is described by two generalised coordinates ϕ and θ as shown in the figure, with $0 \leqslant \theta \leqslant \pi$.

You are given that the total kinetic energy T and potential energy V of the system are

$$T = \frac{1}{4}Ma^2 \dot{\phi}^2 + \frac{1}{2}m \dot{\phi}^2 a^2 \sin^2\theta + \frac{1}{2}ma^2 \dot{\theta}^2,$$
$$V = -mga\cos\theta,$$

where a dot indicates a derivative with respect to time.

Use Lagrange's equations to show that

$$\dot{\phi}\left(M + 2m\sin^2\theta\right) \quad \text{and} \quad \frac{1}{4}a^2\left(M\dot{\phi}^2 + 2m\dot{\phi}^2\sin^2\theta + 2m\dot{\theta}^2\right) - mga\cos\theta$$

are both conserved quantities.

Given the initial state of the system is $\theta = \pi/2$, $\dot{\theta} = 0$ and $\dot{\phi} = \omega$, show that the angular velocity of the wire is

$$\dot{\phi} = \frac{(M + 2m)\omega}{(M + 2m\sin^2\theta)}.$$

Further, show that the bead does not reach the bottom of the wire ($\theta = 0$) if

$$\omega^2 > \frac{2gM}{a(M + 2m)}.$$