UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B8: Mathematics For Physics And Astronomy

COURSE CODE	:	**MATHB008**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0899-3-90

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find the radius of convergence of the following series

$$\text{i)} \sum_{n=0}^{\infty} \left(\frac{z}{2+i}\right)^n \qquad \text{ii)} \sum_{n=0}^{\infty} \frac{(2n)!z^n}{n!^2}, \qquad \text{iii)} \sum_{n=0}^{\infty} \frac{z^n}{n^n}.$$

(b) State de Moivre's theorem and use it to show that

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}.$$

(c) Show

$$\tan^{-1} z = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right),$$

and verify that this formula gives the expected value for $\tan^{-1} 1$.

2. (a) If $w(z) = u(x,y) + iv(x,y)$ is an analytic function of $z = x + iy$ show that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

are satisfied and use them to show that both $u(x,y)$ and $v(x,y)$ are harmonic.

(b) The function $u(x,y) = x^4 + \lambda y^2 x^2 + \mu y^4$ is the real part of an analytic function $f(z)$. Find the values of λ μ, the imaginary part of $f(z)$ in terms of x and y and $f(z)$ itself.

3. (a) Write down Cauchy's integral theorem, integral formula and integral formula for derivatives.

(b) Identify the singularities in the function

$$f(z) = \frac{2z-1}{z(z^2-4)}$$

and show their positions in the Argand diagram. Find the Laurent series for $f(z)$ valid for $|z| > 2$.

(c) Hence or otherwise find $\oint_C f(z)\,dz$ where C is the closed rectangular contour joining the four points $-3+i$, $3+i$, $3-i$ and $-3-i$ in that order.

4. Use suitable contours in the complex plane and the residue theorem to show that

$$\int_0^{2\pi} \frac{d\theta}{3+2\sin\theta} = \frac{2\pi}{\sqrt{5}}, \qquad \int_{-\infty}^{\infty} \frac{dx}{1+x^8} = \frac{\pi}{2}\left(\sin\frac{\pi}{8} + \sin\frac{3\pi}{8}\right).$$

5. (a) The Euler-Lagrange equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

is satisfied by the extremal of the functional

$$\int_a^b F(y, y')dx,$$

in which F has *no explicit dependence on* x, a prime denotes differentiation with respect to x, and $y(a)$ and $y(b)$ are prescribed. Show that

$$F - y'\frac{\partial F}{\partial y'}$$

is constant.

(b) Find a complete solution of the system of Euler-Lagrange equations corresponding to the integral

$$I = \int_{-\pi}^{\pi} 4xy - 6x^2 - 2y^2 + 2\dot{x}^2 + \dot{y}^2\ dt.$$

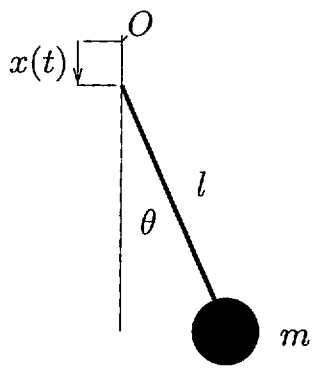
6. Find the extremal curve for the integral

$$I = \int_0^1 (y')^2 + 4y \, dx$$

among those satisfying $y(0) = y(1) = 0$ and $\int_0^1 y \, dx = 4$. Show that the extreme value of I is 208.

7. (a) Define the Lagrangian for a conservative system. Write down Lagrange's equations if the system has n degrees of freedom. State what is meant by an ignorable coordinate and show that whenever such a coordinate exists a corresponding conservation law exists.

   (b) A pendulum bob of mass m is suspended by a light rod of length l from point of support at vertical displacement $x(t)$ beneath a fixed point O, where $x(t)$ is a prescribed function of time, t. Let θ be the angle that the rod makes with the vertical.



Show that

$$V = \text{constant} - mg(x + l\cos\theta)$$

and

$$T = \frac{m}{2}\left(\dot{x}^2 + l^2\dot{\theta}^2 - 2l\sin\theta\dot{x}\dot{\theta}\right)$$

where V and T are the potential and kinetic energies of the system and g is the acceleration due to gravity.

Use Lagrange's equations to show that

$$l\ddot{\theta} - \ddot{x}\sin\theta + g\sin\theta = 0.$$

Simplify this equation on the assumption that θ is small and comment on the case where $x(t)$ is a quadratic function of t. What qualitative change in the motion occurs if $\ddot{x} > g$?