# UNIVERSITY COLLEGE LONDON

*University of London*

## EXAMINATION FOR INTERNAL STUDENTS

*For the following qualifications :-*

*B.Sc.* *M.Sci.*

**Mathematics B8: Mathematics For Physics And Astronomy**

| | | |
|---|---|---|
| COURSE CODE | : | **MATHB008** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **29-APR-02** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 hours** |

02-C0898-3-90



**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **five** *questions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Find the radius of convergence of the power series

$$\text{(i)} \sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}, \qquad \text{(ii)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n^n}.$$

(b) If $z$ is a complex number, show that $(z + z^{-1})(z - z^{-1}) = z^2 + z^{-2}$. If now $z = \cos\theta + i\sin\theta$, show by expanding $(z + z^{-1})^5(z - z^{-1})^5$ that

$$\sin^5\theta\cos^5\theta = \frac{1}{2^9}(\sin 10\theta - 5\sin 6\theta + 10\sin 2\theta).$$

(c) Write down an expression for $\tan z$ in terms of $\exp(iz)$ and use it to find all values of $\tan^{-1}(2i)$.

2. (a) Explain what is meant by the statement "$f(z)$ is an analytic function in the region $D$ of the complex plane".

(b) If $u(x, y)$ and $v(x, y)$ are respectively the real and imaginary parts of an analytic function $f(z)$ with $z = x + iy$, write down the Cauchy-Riemann equations which are satisfied at all points in $D$ and deduce that $u$ and $v$ are harmonic throughout $D$.

(c) Show that the analytic function $f(z) = \exp(z)$ has $u(x, y) = \exp(x)\cos y$ as its real part and $v(x, y) = \exp(x)\sin y$ as its imaginary part. Verify the Cauchy-Riemann equations hold everywhere.

(d) Verify that $h(x, y) = \exp(x)(x\cos y - (y + 1)\sin y)$ is harmonic for all $x$ and $y$ and determine an analytic function $g(z)$ whose imaginary part is $h(x, y)$.

3. (a) Write down Cauchy's integral theorem, integral formula and integral formula for derivatives.

(b) Find values of $A, B, C$ and $D$ such that

$$f(z) = \frac{z^3 - 1}{z(z^2 - 1)} = A + \frac{B}{z} + \frac{C}{z-1} + \frac{D}{z+1}$$

and comment on the result. Find all terms in the Laurent expansions for $f(z)$ in powers of $z$ valid for $|z| < 1$ and for $|z| > 1$.

(c) Using the results of part (b) or otherwise, find the values of

$$\oint_{C_i} \frac{z^3 - 1}{z(z^2 - 1)} dz,$$

where $C_1$ is the circle $|z| = 1/2$ and $C_2$ the circle $|z| = 2$.

4. Use the residue theorem and suitable contours in the complex plane to show

$$\text{(i)} \int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta} = \frac{2\pi}{3}, \qquad \text{(ii)} \int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 - 2x + 2} = \frac{\pi \cos 1}{e}.$$

5. (a) Euler's equation,

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0,$$

is satisfied by the extremal of the functional

$$\int_a^b F(y, y') dx,$$

in which $F$ has *no explicit dependence on* $x$, a prime denotes differentiation with respect to $x$ and $y(a)$ and $y(b)$ are prescribed. Show that

$$F - y' \frac{\partial F}{\partial y'}$$

is constant.

(b) Find the extremal curve of

$$I = \int_0^{\pi/2} \left(y^2 - (y')^2\right) \, dx$$

if $y(0) = 0$, $y(\pi/2) = 1$. Show that $I = 0$ on this extremal.

6. The curve $y = y(x)$ is required so that the area

$$I = \int_0^1 y\, dx, \qquad y \geq 0,$$

is maximised for a given constant value of the length

$$L = \int_0^1 \sqrt{1 + (dy/dx)^2}\, dx,$$

and subject to the end conditions $y(0) = y(1) = 0$. Show that the extremal satisfies the differential equation

$$\left(\frac{dy}{dx}\right)^2 = \frac{\lambda^2}{(y + c)^2} - 1,$$

for constants $\lambda$ and $c$. Show that the solution is a circular arc and derive an equation relating $L$ and the radius of the circular arc.

7. (a) Define the Lagrangian for a conservative system. Write down Lagrange's equations if the system has $n$ degrees of freedom.

(b) A double pendulum consists of two masses $m$ joined by a light rod $AB$ with the mass at $A$ joined to a fixed point $O$ by a light rod $OA$. The rods have equal length $l$ and are freely hinged at $O$ and $A$ so that they may move in a vertical plane. If $\theta_1$ and $\theta_2$ are the inclinations of $OA$ and $OB$ to the downward vertical then *you are given* that the potential energy $V$ and kinetic energy $T$ of the pendulum are

$$V = -mgl(2\cos\theta_1 + \cos\theta_2),$$

$$T = \frac{ml^2}{2}(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)),$$

where $g$ is the acceleration due to gravity. Show that Lagrange's equations are

$$2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^2\sin(\theta_1 - \theta_2) = -2(g/l)\sin\theta_1,$$

$$\ddot{\theta}_1\cos(\theta_1 - \theta_2) + \ddot{\theta}_2 - \dot{\theta}_1^2\sin(\theta_1 - \theta_2) = -(g/l)\sin\theta_2.$$

If the pendulum performs small oscillations about the downward vertical, show that the frequencies, $\omega$, of the normal modes of oscillation satisfy

$$\omega^2 = (2 \pm \sqrt{2})(g/l).$$