UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B6: Mathematical Methods In Chemistry

COURSE CODE	:	**MATHB006**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0906-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Verify that one solution of the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x(x+1)\frac{dy}{dx} + 2(x+1)y = 0, \qquad (1)$$

is $y = x$.

By substituting $y = xu(x)$ into equation (1), show that $u(x)$ satisfies a second-order differential equation with constant coefficients. Hence find a second linearly independent solution to (1).

Substituting the series $y = \sum_{n=0}^{\infty} a_n x^n$ into (1), show that $a_0 = 0$ and a_1 is arbitrary. Derive a recurrence relation for the coefficients a_n for $n > 2$. Deduce that

$$a_n = \frac{C2^{n-1}}{(n-1)!},$$

where C is a constant. Show that the series solution is the same as the solution found above.

2. Show that $x = 0$ is a regular singular point of the differential equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0.$$

Assuming a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+r},$$

show that the roots of the indicial equation differ by an integer. For the larger of these roots show that $a_1 = 0$ and find the recurrence relation satisfied by the coefficients a_n. Deduce that $a_{2n+1} = 0$ and

$$a_{2n} = \frac{(-1)^n a_0}{(2n+1)!}, \quad n \geq 1.$$

Hence show that one solution of the differential equation is

$$y = a_0\frac{\sin x}{x^{1/2}}.$$

3. (a) Write down the generating function for Legendre polynomials $P_n(x)$. Use it to show

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x).$$

Given that $P_0(x) = 1$ and $P_1(x) = x$, find $P_3(x)$.

(b) Given that $y = P_n(x)$ satisfies

$$\left[(1-x^2)y'\right]' + n(n+1)y = 0,$$

where $n \geq 1$ is a positive integer and the dashes denote derivatives, show that

$$\left[(1-x^2)(P_n'P_m - P_nP_m')\right]' + [n(n+1) - m(m+1)]P_nP_m = 0.$$

Deduce that, if $n \neq m$,

$$\int_{-1}^{1} P_nP_m dx = 0.$$

4. In spherical polar coordinates Laplace's equation for $u(r, \theta, \phi)$ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0.$$

To what does this equation reduce if axisymmetric solutions $u(r, \theta)$ are sought?

Show, by directly substituting

$$u = r^m P_n(\cos\theta)$$

into the axisymmetric version of Laplace's equation that m has either of the values n or $-n-1$ and $P_n(\mu)$ satisfies Legendre's equation

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + n(n+1)P_n = 0.$$

The function u is to be found in the region outside the sphere $r = a$ and u must tend to zero as $r \to \infty$. If $u = f(\theta)$ on $r = a$, show that

$$u = \sum_{n=0}^{\infty} A_n \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta),$$

where

$$A_n = \frac{2n+1}{n}\int_0^{\pi} f(\theta)P_n(\cos\theta)\sin\theta d\theta.$$

If $f(\theta) = \sin\theta + \cos\theta$ determine the constant A_0.

You may assume the orthogonality property of the Legendre polynomials

$$\int_{-1}^{1} P_n(\mu)P_m(\mu)d\mu = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n. \end{cases}$$

5. (a) Define a group. Let X be a fixed element of a group G with inverse X^{-1}. Consider the set of elements of the form XAX^{-1}, where A is any element of G, and the law of multiplying the elements to form the product XAX^{-1} is the same as the law of multiplication defining G. Show that this set is also a group under the same multiplication law.

   (b) A cyclic group H of order 4 has elements I, A, B, C, where I is the identity element. Write down the group multiplication table for H. Another group H' has elements I', A', B', C', where I' is the identity element, is not isomorphic to H. Write down its group multiplication table.

6. (a) If λ is an eigenvalue of an invertible matrix A with corresponding eigenvector $\mathbf{x}$, show that λ^{-1} is an eigenvalue of the matrix A^{-1} with eigenvector $\mathbf{x}$.

   (b) Define a unitary matrix U. If $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of a unitary matrix corresponding to distinct eigenvalues λ and μ respectively, show that $\bar{\mathbf{u}}^T\mathbf{v} = 0$.
   [*You may assume that the eigenvalues of a unitary matrix have unit modulus.*]

7. (a) The motion of two linked particles with displacements x_1 and x_2 is governed by

$$\ddot{\mathbf{x}} = A\mathbf{x},$$

   where A is a 2×2 matrix and $\mathbf{x}$ is a column vector with components x_1 and x_2. State the condition on the eigenvalues of A in order for the motion of both particles to be oscillatory.

   (b) Solve the coupled system

$$\ddot{x}_1 + 2x_1 - x_2 = 0,$$

$$\ddot{x}_2 - x_1 + 2x_2 = 0,$$

   for two particles released from rest with $x_1 = 3$ and $x_2 = 1$ at $t = 0$.

END OF PAPER