UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics B6: Mathematical Methods In Chemistry

COURSE CODE	:	**MATHB006**
UNIT VALUE	:	**0.50**
DATE	:	**27-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0898-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **five** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find the general solution of the differential equation

$$y'' - 5y' + 6y = 2e^{2x} + 6x^2.$$

   (b) Using the substitution $x = t^3$, convert the differential equation

$$9x^2y'' + 6xy' + \lambda^2 x^{2/3} y = 0,$$

   where $\lambda \neq 0$ is real, into one with constant coefficients. Non-zero solutions to the differential equation are sought which satisfy $y = 0$ when $x = 0$ and $x = 1$. Show that such solutions are possible only if $\lambda = n\pi$, $n = \pm 1, \pm 2, \pm 3, \ldots$.

2. Chebyshev's equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + k^2 y = 0, \quad -1 < x < 1,$$

where k is a real constant.

By substituting the series $y = \sum_{m=0}^{\infty} a_m x^m$ into Chebyshev's equation, show that

$$a_{m+2} = \frac{m^2 - k^2}{(m+2)(m+1)} a_m,$$

for all $m \geq 0$. If $k = n$ is a positive integer, explain why there is a polynomial solution of degree n, $T_n(x)$, to Chebyshev's equation.

Use the substitution $x = \cos\theta$, $0 < \theta < \pi$, to show that Chebyshev's equation transforms into a second order ODE with constant coefficients. Find the general solution of this equation.

The Chebyshev polynomial solutions, $T_n(x)$, are defined as

$$T_n(x) = \cos[n \cos^{-1} x], \quad n = 0, 1, 2, \ldots.$$

Verify that the Chebyshev polynomials are a class of solutions which belong to the general solution found above.

3. (a) Legendre's differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \alpha(\alpha+1)y = 0,$$

where α is a constant. When $\alpha = n$ is a positive integer, Legendre's equation has a polynomial solution $P_n(x)$.
Let $P_n(x)$ and $P_m(x)$ be two polynomial solutions of Legendre's equation with $n \neq m$. Starting from Legendre's equation, show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0.$$

(b) Given that

$$\int_{-1}^{1} (P_n(x))^2\,dx = \frac{2}{2n+1},$$

show that if a function $f(x)$ is written as a series of Legendre polynomials i.e.

$$f(x) = \sum_{k=0}^{\infty} \alpha_k P_k(x),$$

then the coefficients α_k, $k = 0, 1, 2, \ldots$, are given by

$$\alpha_k = \frac{2k+1}{2}\int_{-1}^{1} f(x)P_k(x)dx.$$

(c) Find the first three terms in the Legendre polynomial expansion of $y = |x|$.

[Hint: You may assume $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = (3x^2-1)/2$.]

4. (a) Write down the generating function for Legendre polynomials $P_n(x)$. Use it to show $P_n(-x) = (-1)^n P_n(x)$.

   (b) In plane polar coordinates, P is the point $(a, 0)$ and Q is the point (r, θ). If R is the distance PQ and $a < r$ then show

$$\frac{1}{R} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{a}{r}\right)^n P_n(\cos\theta).$$

   (c) Two electric charges of strength $+e$ and $-e$ are located at $(+a, 0)$ and $(-a, 0)$. A point S is at distance r from the midpoint O of the line joining the two charges. The line OS makes an angle θ with the line joining the two charges. Find the electric potential V at S in terms of an infinite series of Legendre polynomials. Show that the limiting value of V as $a \to 0$ and $e \to \infty$ such that ea remains constant and equal to q is

$$V = \frac{2q}{r^2} P_1(\cos\theta).$$

   [Hint: the electric potential V due to a point charge e at a distance R from the charge is given by $V = e/R$.]

5. (a) Define a group. Explain what is meant by saying two groups are isomorphic.

   (b) Consider the set of all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix},$$

   where $a \neq 0$ is a real number. Show that the set forms a group under matrix multiplication. Find, giving reasons, another group which is isomorphic to this group.

6. (a) Define what is meant by (i) a skew-Hermitian matrix, and (ii) a unitary matrix.

   (b) For any invertible matrix C show that $(C^T)^{-1} = (C^{-1})^T$.

   (c) Prove that if A is skew-Hermitian then $(I - A)(I + A)^{-1}$ is unitary (assuming $I + A$ is invertible).

7. The motion of two linked particles is governed by the pair of differential equations

$$\ddot{x}_1 + 2x_1 - x_2 = 0,$$

$$\ddot{x}_2 - x_1 + 2x_2 = 0.$$

Find the general solution for this system.

If the system is released from rest with $x_1 = 2$ and $x_2 = 0$ at $t = 0$, show that

$$x_1 = \cos t + \cos \sqrt{3}t.$$

Find also x_2.