UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M242: Mathematical Methods 4

COURSE CODE	:	**MATHM242**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1051-3-80

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Use the method of separation of variables to show that one solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \le x \le l, \quad 0 \le y \le h,$$

$$u(0,y) = 0, \quad lu_x(l,y) + u(l,y) = 0, \quad u_y(x,0) = 0,$$

is of the form $u(x,y) = X(x)Y(y)$, where

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad X(0) = 0, \quad l\frac{dX}{dx}(l) + X(l) = 0,$$

$$\frac{d^2Y}{dy^2} - \lambda^2 Y = 0, \quad \frac{dY}{dy}(0) = 0.$$

Explain how an infinite number of suitable values for λ may be found. Find an equation satisfied by these values, λ_n, and show that the general solution is

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\cosh(\lambda_n y).$$

Find integral expressions for A_n if in addition $u(h,x) = f(x)$ and show that

$$u(x,0) = \frac{2}{l}\sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)\int_0^l f(q)\sin\lambda_n q\, dq}{\cosh(\lambda_n h)(1+\cos^2(\lambda_n l))}.$$

2. Show that the differential equation, satisfied by the function $y(x)$,

$$x^2y'' + (x^2 - x)y' + (1+x)y = 0,$$

where a prime denotes differentiation with respect to x, has a regular singular point at $x = 0$.

Show that one solution to the equation is $y(x) = x^c \sum_{k=0}^{\infty} a_k x^k$, where c takes a value to be determined, $a_0 = 1$ and

$$a_k = (-1)^k \frac{(k+c)}{(k+c-1)(k+c-2)\cdots(2+c)(1+c)c^2}.$$

Show that a second, independent, solution of the equation is

$$y = x\ln x \sum_{k=0}^{\infty} \frac{(-1)^k(k+1)}{k!}x^k - x\sum_{k=1}^{\infty} \frac{(-1)^k(k+1)}{k!}\left[\frac{k}{k+1} + \sum_{r=1}^{r=k}\frac{1}{r}\right]x^k.$$

3. The generating function for the Legendre polynomials $P_n(x)$ is

$$\frac{1}{(1-2\omega x+\omega^2)^{1/2}} = \sum_{n=0}^{n=\infty} \omega^n P_n(x).$$

(a) Show that

$$(n+1)P_{n+1}(x) = x(2n+1)P_n(x) - nP_{n-1}(x), \quad n \geq 1.$$

(b) Use the result

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0, \quad n \neq m,$$

to show

$$\frac{1}{\omega}\left[\ln(1+\omega) - \ln(1+\omega)\right] = \sum_{n=0}^{\infty} \omega^{2n} \int_{-1}^{1} P_n^2(x)\, dx.$$

Deduce that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

4. A function $f(x)$ and its Fourier transform, $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\, dk.$$

(a) Show $\widehat{(f * g)} = \sqrt{2\pi}\hat{f}\hat{g}$, where

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy.$$

(b) Find the solution $h(x)$ to the integral equation

$$\int_{-\infty}^{\infty} \frac{h(u)du}{a^2+(x-u)^2} = \frac{1}{b^2+x^2}, \quad 0 < a < b.$$

[You may assume the result $\int_0^{\infty} \frac{\cos(px)dx}{x^2+q^2} = (\pi/2q)\exp(-|p|q)$ but you should derive all other results that you use.]

5. Consider the equation, satisfied by the function $u(x,t)$

$$u_t = u_{xx}, \quad -\infty < x < \infty, \quad t \geq 0, \quad u(x,0) = f(x),$$

where both $f(x)$ and $u(x,t)$ tend to zero as $|x| \to \infty$.
Use Fourier transforms to show that

$$u(x,t) = \int_{-\infty}^{\infty} f(x-q)h(q,t)dq,$$

and find $h(x,t)$. Show that if $f(x) = \exp(-x^2)$,

$$u(x,t) = \frac{\exp(-x^2/(1+4t))}{\sqrt{1+4t}}.$$

[You may assume the result $\int_{-\infty}^{\infty} \exp(-a(x+b)^2)dx = \sqrt{\pi/a}$], $a > 0$ but you should derive all other results that you use.]

6. The Laplace transform $\mathcal{L}[f](s) = \bar{f}(s)$ of a function $f(t)$ is defined by

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t)\, dt.$$

(a) Use this definition to show that

$$\text{(i) } \mathcal{L}[e^{-at}] = 1/(s+a), \quad \text{(ii) } \mathcal{L}[df/dt] = s\bar{f}(s) - f(0).$$

(b) If $u(x,t)$ satisfies the partial differential equation, initial and boundary conditions

$$u_t + xu_x = x^n, \qquad u(x,0) = x^m, \quad u(0,t) = 0, \quad m, n > 0,$$

show that

$$\bar{u}(x,s) = \frac{x^m}{s+m} + \frac{x^n}{s(s+n)},$$

where $\bar{u}(x,s)$ is the Laplace transform in t of $u(x,t)$. Hence find $u(x,t)$.

END OF PAPER