UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M242: Mathematical Methods 4

COURSE CODE : **MATHM242**

UNIT VALUE : **0.50**

DATE : **26-MAY-05**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

05-C0969-3-110

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. The function $u(r,\theta)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0,$$

in the regions $r\le a$, $0\le\theta\le\pi$. Show that solutions of the type

$$u(r,\theta)=R(r)w(\cos\theta)$$

are possible if

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\lambda(\lambda+1)w=0, \qquad (1)$$

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-\lambda(\lambda+1)R=0,$$

for constant λ.

You are given that the differential equation (1) has solutions regular at $z=\pm1$ only if $\lambda(\lambda+1)=n(n+1)$, $n=0,1,2,\ldots$, and that the solution in this case, normalised so that $w(1)=1$, is $w=P_n(z)$ with $P_0(z)=1$, $P_1(z)=z$, $P_2(z)=(3z^2-1)/2$.

Deduce that, if $u(r,\theta)$ is regular everywhere,

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta).$$

If u satisfies the boundary condition

$$u(a,\theta)=\alpha+\beta\cos\theta+\gamma\cos^2\theta,$$

then find u when $r=0$.

2. Show that the differential equation

$$x^2y'' - xy' + (1+x)y = 0,$$

where a prime indicates differentiation with respect to x, has a regular singular point at $x = 0$.

Show that one solution to the equation is $y_1(x) = x^{\lambda}\sum_{k=0}^{\infty} a_k x^k$, $a_0 = 1$, where λ takes a value to be determined and

$$a_k = \frac{(-1)^k}{(k+\lambda-1)^2(k+\lambda-2)^2\ldots(\lambda+1)^2\lambda^2}.$$

Show that a second, independent, solution of the equation is

$$y_2(x) = x\ln x\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{(k!)^2} - 2x\sum_{k=1}^{\infty}\frac{(-1)^k x^k S_k}{(k!)^2}, \qquad S_k = \sum_{r=1}^{k}\frac{1}{r}.$$

3. (a) Given $y = J_n(\lambda x)$ satisfies the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0, \quad n = 0, 1, 2, \ldots,$$

verify that

$$\int_0^1 xJ_n(\lambda x)J_n(\lambda_i x)\,dx = \frac{\lambda_i J_n(\lambda)J_n'(\lambda_i)}{\lambda^2 - \lambda_i^2}, \quad (\lambda \neq \lambda_i)$$

if λ_i is a positive root of $J_n(x) = 0$ and a prime denotes differentiation. Deduce that the integral vanishes if λ is a root of $J_n(x) = 0$ other than λ_i. Find an expression for

$$\int_0^1 xJ_n(\lambda_i x)^2\,dx.$$

(b) The generating function for the Laguerre Polynomials $L_n(x)$, $n = 0, 1, 2, \cdots$, is

$$\frac{1}{1-t}\exp\left[-\frac{xt}{1-t}\right] = \sum_{n=0}^{n=\infty} t^n L_n(x).$$

Use this to show that

(i) $L_0(x) = 1$,

(ii) $L_n(0) = 1$, $n = 0, 1, 2, \cdots$,

(iii) $L_{n+1}' = L_n' - L_n$, where a prime denotes differentiation. Find $L_1(x)$ and $L_2(x)$.

4. (a) Write down the definition of the Fourier sine transform and its inversion formula.

   (b) Use the Fourier sine transform to show that the solution of $\nabla^2\phi = 0$ in the quarter plane $x \geq 0$, $y \geq 0$ with $\phi(x,y) \to 0$ as $x^2 + y^2 \to \infty$, $\phi(0,y) = 0$ and $\phi(x,0) = f(x)$, $f(x) \to 0$ as $x \to \infty$, may be written as

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \int_0^\infty f(u)\exp(-ky)\sin(kx)\sin(ku)\,dk\,du.$$

   (c) Show that

$$\phi(x,y) = \frac{y}{\pi}\int_0^\infty f(u)\left[\frac{1}{y^2+(x-u)^2} - \frac{1}{y^2+(x+u)^2}\right]du.$$

5. A function $f(x)$ and its Fourier transform, $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{-ikx}\,dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(k)e^{ikx}\,dk.$$

   (a) Show

   (i)

$$f(0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \hat{f}(k)\,dk,$$

   (ii)

$$\widehat{(f*g)} = \sqrt{2\pi}\hat{f}\hat{g},$$

   where

$$(f*g)(x) = \int_{-\infty}^\infty f(x-y)g(y)\,dy,$$

   (iii)

$$\widehat{f(-x)} = \hat{f}^*$$

   where $*$ represents the complex conjugate,

   (iv)

$$\int_{-\infty}^\infty |\hat{g}(k)|^2\,dk = \int_{-\infty}^\infty g(y)^2\,dy.$$

   (b) Verify the last result of section (a) if

$$g(y) = \begin{cases} 0 & y < 0 \\ \exp(-y) & y \geq 0. \end{cases}$$

6. (a) State the definition of the Laplace transform $\mathcal{L}[h(t)](s)$ for a function $h(t)$, and use it to prove

$$\mathcal{L}[1] = 1/s, \quad \mathcal{L}[\exp(-at)] = \frac{1}{s+a}$$

$$\mathcal{L}\left[\frac{df}{dt}\right] = s\mathcal{L}[f(t)] - f(0),$$

where $a \geq 0$, $f(t) = 0$ for $t < 0$.

(b) Write down the convolution theorem for the Laplace transform and use it to show

$$\mathcal{L}\left[\int_0^t f(q)\, dq\right] = \frac{1}{s}\mathcal{L}[f(t)].$$

(c) Show that the Laplace transform of the solution to the equation

$$\frac{dy}{dt} + 6y + 5\int_0^t y(q)\, dq = \begin{cases} 1 & 0 \leq t \leq 1, \\ 0 & t > 1, \end{cases} \qquad y(0) = 0,$$

for $t \geq 0$, is

$$\mathcal{L}[y] = \frac{1 - \exp(-s)}{(s+1)(s+5)}.$$

Find $y(t)$.

END OF PAPER