UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M242: Mathematical Methods 4

COURSE CODE : **MATHM242**

UNIT VALUE : **0.50**

DATE : **28-APR-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0963-3-120

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Use the method of separation of variables to show that the solution of the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial u}{\partial x}\right], \quad -1 \le x \le 1, \quad t > 0, \quad u(x,0) = f(x),$$

that is both analytic for $-1 \le x \le 1$ and bounded as $t \to \infty$, is

$$u(x,t) = \sum_{n=0}^{n=\infty} \exp(-n(n+1)t) A_n P_n(x).$$

Here $P_n(x)$ is the Legendre polynomial of order n and A_n are constants. Find expressions for A_n in terms of $f(x)$. Comment on the solution as $t \to \infty$.

[You may use the two results below without proof, but you should state clearly any other results you use.

(a) $y = P_n(x)$ is the polynomial solution of Legendre's equation of order n, $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

(b) $\int_{-1}^{1} P_n^2(x)dx = 2/(2n+1)$.]

2. Show that the differential equation

$$xy'' + (1+x)y' + 2y = 0,$$

where a prime denotes differentiation with respect to x, has a regular singular point at $x = 0$.

Show that one solution to the equation is $y(x) = x^c \sum_{k=0}^{\infty} a_k x^k$, where c takes a value to be determined, $a_0 = 1$ and

$$a_k = (-1)^k \frac{(k+c+1)}{(k+c)(k+c-1)\cdots(2+c)(1+c)^2}.$$

Show that a second, independent, solution of the equation is

$$y = \ln x \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{k!} x^k + \sum_{k=1}^{\infty} (-1)^k \left[\frac{1}{k!} - (1+S_k)\frac{k+1}{k!}\right] x^k, \qquad S_k = \sum_{r=1}^{r=k} \frac{1}{r}.$$

3. (a) Legendre's equation of order n is

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \quad n = 0, 1, 2, \cdots,$$

where a prime denotes differentiation with respect to x. *You are given* that it has a polynomial solution $P_n(x)$. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)\, dx = 0, \quad n \neq m.$$

(b) The generating function for the Bessel functions $J_n(x)$ is

$$\exp\left[x(\omega - \omega^{-1})/2\right] = \sum_{n=-\infty}^{n=\infty} \omega^n J_n(x).$$

Use this to show that

(i) $J_n(x) = J_{-n}(-x)$,

(ii) $J_n(x) = (-1)^n J_{-n}(x)$,

(iii)

$$\exp(ix\sin\theta) = J_0(x) + 2i \sum_{n \text{ odd}} J_n(x)\sin n\theta + 2 \sum_{n \text{ even}} J_n(x)\cos n\theta$$

Use the result (iii) to show that, if m is an integer,

$$\int_{-\pi}^{\pi} \cos m\theta \cos(x\sin\theta)\, d\theta = \begin{cases} 2\pi J_m(x) & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

4. A function $f(x)$ and its Fourier transform, $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\, dk.$$

(a) Show

(i) $\widehat{f'(x)} = ik\hat{f}(k)$,

(ii) $\widehat{(f * g)} = \sqrt{2\pi}\hat{f}\hat{g}$, where $(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy$.

(b) Consider the equation for $u(x,y)$, $-\infty < x < \infty$, $y > 0$

$$\nabla^2 u = 0, \quad u(x,0) = f(x), \quad u(x,y) \to 0 \text{ as } \sqrt{x^2+y^2} \to \infty.$$

Use the Fourier transform to show that

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(s)}{y^2 + (x-s)^2} ds.$$

If

$$f(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1, \end{cases}$$

then show that

$$u(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1-x}{y}\right) + \tan^{-1}\left(\frac{1+x}{y}\right)\right].$$

5. (a) State the definition of the Laplace transform $\mathcal{L}[h(t)](s)$ for a function $h(t)$, and use it to prove

$$\mathcal{L}\,[\mathrm{H}(t)] = 1/s, \quad \mathcal{L}\,[f(t-a)] = \exp(-as)\mathcal{L}[f(t)],$$

$$\mathcal{L}\,[tf(t)] = -\frac{d}{ds}\mathcal{L}[f(t)],$$

where $a \geq 0$, $f(t) = 0$ for $t < 0$ and $\mathrm{H}(t)$ is the Heaviside step function.

(b) Use the Laplace transform to show that the solution to the equation

$$\frac{d^2x}{dt^2} + x(t) = t\,\mathrm{H}(t-a), \quad x(0) = 0, \quad \frac{dx}{dt}(0) = 0,$$

for $a,\ t \geq 0$ is

$$x(t) = \begin{cases} 0 & \text{if } \ t < a, \\ t - a\cos(t-a) - \sin(t-a) & \text{if } \ t \geq a. \end{cases}$$

[You may use the Bromwich inversion formula

$$x(t) = \frac{1}{2\pi i}\int_\gamma e^{st}\mathcal{L}[x](s)\,ds,$$

with a suitable choice of γ]

6. Consider the integral equation

$$u(x) + \int_{-\infty}^{\infty} f(x-\xi)u(\xi)\,d\xi = g(x),$$

where f and g are given functions and u is to be found. Show that if $\hat{f}(k)$ and $\hat{g}(k)$ are the Fourier transforms of f and g respectively, then

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\hat{g}(k)}{1+\sqrt{2\pi}\hat{f}(k)}e^{ikx}\,dk.$$

Show that if

$$g(x) = e^{-x^2/2}, \quad f(x) = \tfrac{1}{2}e^{-|x|}.$$

then

$$u(x) = e^{-x^2/2} - \frac{1}{2\sqrt{2}}\int_{-\infty}^{\infty} e^{-s^2/2}e^{-\sqrt{2}|x-s|}\,ds.$$

[You may use the results

$$\int_{-\infty}^{\infty}\frac{\cos kx\,dk}{a^2+k^2} = \frac{\pi}{a}e^{-a|x|}, \ a > 0, \quad \int_{-\infty}^{\infty} e^{-x^2/2}\cos kx\,dx = \sqrt{2\pi}e^{-k^2/2}.]$$