UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M242: Mathematical Methods 4

COURSE CODE	:	**MATHM242**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0956-3-90

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. The function $u(r,\theta)$ satisfies the Laplace equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0,$$

in the region $r\ge a,\ 0\le\theta\le\pi/2$ where r and θ are the usual polar coordinates. The boundary conditions

$$u(r,0)=0,\quad u(r,\pi/2)+\frac{\partial u}{\partial\theta}(r,\pi/2)=0,\quad u(a,\theta)=f(\theta),\quad u(r,\theta)\to 0 \text{ as } r\to\infty,$$

are imposed.

(a) Look for solutions of the type $u(r,\theta)=R(r)\Theta(\theta)$ and show that

$$\Theta''+\lambda^2\Theta=0,$$

$$r^2R''+rR'-\lambda^2R=0,$$

for some $\lambda>0$. Show also that λ must take values $\lambda=\lambda_n,\ n=1,2,\cdots$, where

$$\tan(\lambda_n\pi/2)+\lambda_n=0.$$

(b) Show graphically that there are an infinite number of such λ_n.

(c) Show that

$$\frac{\partial u}{\partial r}(a,\theta)=-\sum_{n=1}^{\infty}\frac{\lambda_n\int_0^{\pi/2}f(\phi)\sin\lambda_n\phi\,d\phi}{a\int_0^{\pi/2}\sin^2\lambda_n\phi\,d\phi}\sin\lambda_n\theta.$$

2. Show that the differential equation

$$(x^2 - x)y'' + (7x - 1)y' + 9y = 0,$$

where a prime denotes differentiation with respect to x, has a regular singular point at $x = 0$.

Show that one solution to the equation is $y(x) = x^c \sum_{k=0}^{\infty} a_k x^k$, where c takes a value to be determined and

$$a_k = \frac{(k+c+2)^2}{(k+c)^2} a_{k-1}.$$

Show that a second, independent, solution of the equation is

$$y = \frac{1}{4} \ln x \sum_{k=0}^{\infty} (k+2)^2 (k+1)^2 x^k - \frac{3}{4} \sum_{k=1}^{\infty} (k+2)(k+1)k(k+5/3)x^k.$$

3. Legendre's equation of order n is

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0, \quad n = 0, 1, 2, \cdots,$$

where a prime denotes differentiation with respect to x. *You are given* that it has a polynomial solution $P_n(x)$ where

$$\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}.$$

(a) Show

$$P_n(1) = 1, \quad P_{2m+1}(0) = 0, \quad P_{2m}(0) = (-1)^m (2m)!/(4^m (m!)^2), \quad m = 0, 1, 2, \cdots.$$

(b) By considering $h(x) = (1 - x^2)^n$, or otherwise, show that

$$v_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

satisfies Legendre's equation of order n.

(c) Show $v_n(1) = 1$ and deduce $v_n(x) = P_n(x)$.

4. A function $f(x)$ and its Fourier transform, $\hat{f}(k)$ are related through

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx \quad \text{and} \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx}\, dk.$$

(a) Show

$$(\widehat{f * g}) = \sqrt{2\pi} \hat{f} \hat{g}, \quad \text{where} \quad (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y)\, dy.$$

and deduce

$$\int_{-\infty}^{\infty} \hat{f}(k) \hat{g}(k)\, dk = \int_{-\infty}^{\infty} f(-x) g(x)\, dx.$$

(b) If $h(x) = 1/(x^2 + a^2)$, $a > 0$, show that

$$\hat{h}(k) = \sqrt{\frac{\pi}{2}} \frac{\exp(-a|k|)}{a},$$

and deduce that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{ab(a+b)}, \quad a > 0, \quad b > 0.$$

5. Consider the Laplace equation for $\phi(x, y)$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y \geq 0,$$

with boundary conditions

$$\phi(x, y) \quad \text{bounded as} \quad \sqrt{x^2 + y^2} \to \infty, \quad \phi(x, 0) = S_\epsilon(x),$$

where

$$S_\epsilon(x) = \operatorname{sgn}(x) \exp(-\epsilon|x|), \quad \epsilon > 0, \quad \operatorname{sgn}(x) = \begin{cases} 1 & x > 0, \\ -1 & x < 0, \\ 0 & x = 0. \end{cases}$$

(a) Use Fourier transforms to show that

$$\phi(x, y) = \frac{2}{\pi} \int_0^{\infty} \frac{k \exp(-ky)}{\epsilon^2 + k^2} \sin kx\, dk.$$

(b) Show that, if $\epsilon = 0$, then

$$\frac{\partial \phi}{\partial x} = \frac{2}{\pi} \frac{y}{x^2 + y^2}.$$

6. The Laplace transform $\mathcal{L}[f](s) = \bar{f}(s)$ of the function $f(t)$ where $f(t) = 0$ for $t < 0$ is defined as

$$\bar{f}(s) = \int_0^\infty \exp(-st) f(t)\, dt.$$

(a) Show

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}, \quad \mathcal{L}[\exp(at)] = \frac{1}{s-a} \quad \text{and} \quad \mathcal{L}\left[t^2 \frac{d^2 f}{dt^2}\right] = \frac{d^2}{ds^2}\left(s^2 \bar{f}(s)\right).$$

(b) The function $y(t)$ satisfies the equation

$$t^2 \frac{d^2 y}{dt^2} - 2y = t^n, \quad n = 0, 1, 2, \cdots.$$

Show that its Laplace transform, $\bar{y}(s)$ satisfies the equation

$$s^2 \frac{d^2 \bar{y}}{ds^2} + 4s \frac{d\bar{y}}{ds} = \frac{n!}{s^{n+1}}.$$

Hence show that a solution for $y(t)$ is given by

$$y(t) = At^2 + \frac{t^n}{(n-2)(n+1)},$$

with A an arbitrary constant.

(c) *Use the Laplace transform* to show that the integral equation

$$y(t) = \exp(-t) + 3\int_0^t \exp(-2z) y(t-z)\, dz,$$

has solution

$$y(t) = \tfrac{3}{2}\exp(t) - \tfrac{1}{2}\exp(-t).$$

You may assume that $\mathcal{L}[\int_0^t g(z)h(t-z)\, dz] = \mathcal{L}[g]\mathcal{L}[h]$.