UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M242: Mathematical Methods 4

COURSE CODE	:	**MATHM242**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0951-3-70

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. The function $u(r,\theta)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0,$$

in the regions $r\geq a$, $0\leq\theta\leq\pi$. Show that solutions of the type

$$u(r,\theta)=r^\lambda w(\cos\theta)$$

are possible if

$$(1-z^2)\frac{d^2w}{dz^2}-2z\frac{dw}{dz}+\lambda(\lambda+1)w=0. \qquad (1)$$

You are given that the differential equation (1) has solutions regular at $z=\pm1$ only if $\lambda(\lambda+1)=n(n+1)$, $n=0,1,2,\ldots$ and that the solution in this case, normalised so that $w(1)=1$, is $w=P_n(z)$. Deduce that, if $u(r,\theta)$ is regular and $u(r,\theta)\to0$ as $r\to\infty$,

$$u(r,\theta)=\sum_{n=0}^{\infty}\frac{A_n}{r^{n+1}}P_n(\cos\theta).$$

Verify that, if $n=0,1,2$, equation (1) has solutions $P_0(z)=1$, $P_1=z$ and $P_2(z)=(3z^2-1)/2$. If u satisfies the boundary condition $u(a,\theta)=\cos^2\theta$, then show that

$$\frac{\partial u}{\partial r}(a,\theta)=\frac{1}{3a}-\frac{3\cos^2\theta}{a}.$$

2. Show that the differential equation

$$x^2y''+2xy'+(\tfrac{1}{4}-x)y=0$$

has a regular singular point at $x=0$.

Show that one solution to the equation is $y_1(x)=x^c\sum_{k=0}^{\infty}a_kx^k$, where c takes a value to be determined, and

$$a_k=\frac{1}{(k+c+\frac{1}{2})^2(k+c-\frac{1}{2})^2(k+c-\frac{3}{2})^2\cdots(c+\frac{3}{2})^2}.$$

Show that a second, independent, solution of the equation is

$$y_2(x)=y_1\ln x-2x^{-1/2}\sum_{k=1}^{\infty}\frac{x^k}{(k!)^2}S_k,\quad S_k=\sum_{n=1}^{n=k}\frac{1}{n}.$$

3. (a) The Bessel functions $J_n(x)$, with $n = 0, \pm 1, \pm 2, \ldots$ satisfy the equation

$$\exp\left[\tfrac{1}{2}x\,(t - 1/t)\right] = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Use this to obtain the following results

(i) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$,

(ii) $2nJ_n(x)/x = J_{n-1}(x) + J_{n+1}(x)$,

(iii) $2\pi J_0(x) = \int_0^{2\pi} \cos(x \sin\theta)\, d\theta$ [*Hint:* put $t = \exp(i\theta)$].

(b) *You are given* that $y = J_0(px)$ satisfies the equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + p^2 x^2 y = 0.$$

Show that

$$\frac{d}{dx}\left\{x^2\left(\frac{dy}{dx}\right)^2\right\} + p^2 x^2 \frac{d}{dx}\left\{y^2\right\} = 0.$$

Hence deduce that if $J_0(pl) = 0$,

$$2\int_0^l x[J_0(px)]^2\, dx = l^2[J_0'(pl)]^2.$$

4. Consider the integral equation

$$f(x) = \int_{-\infty}^{\infty} g(x - \xi) u(\xi)\, d\xi,$$

where f and g are given functions and u is to be found. Show that if $\hat{f}(k)$ and $\hat{g}(k)$ are the Fourier transforms of f and g respectively, then

$$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\hat{f}(k)}{\hat{g}(k)} e^{ikx}\, dk.$$

Hence find $u(x)$ when

$$f(x) = e^{-x^2/2}, \quad g(x) = \tfrac{1}{2}e^{-|x|}.$$

[You may use the results

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\, dx, \qquad \widehat{\frac{d^2 f}{dx^2}} = -k^2 \hat{f}, \qquad \int_{-\infty}^{\infty} e^{-x^2/2}\cos kx\, dx = \sqrt{2\pi}e^{-k^2/2}.]$$

5. (a) The Laplace transform $\mathcal{L}[f](s) = \bar{f}(s)$ of a function $f(t)$ is defined by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Use this definition to show that

(i) $\mathcal{L}[1] = 1/s$, (ii) $\mathcal{L}[e^{-at}] = 1/(s+a)$, (iii) $\mathcal{L}[df/dt] = s\bar{f}(s) - f(0)$.

(b) If $u(x,t)$ satisfies the partial differential equation, initial and boundary conditions

$$u_t + xu_x = x, \qquad u(x,0) = 1 + x^2, \quad u(0,t) = 1,$$

show that

$$\bar{u}(x,s) = \frac{x^2}{s+2} + \frac{x}{s(s+1)} + \frac{1}{s},$$

where $\bar{u}(x,s)$ is the Laplace transform in t of $u(x,t)$. Hence find $u(x,t)$.

6. Give the definition of the generalised Delta function δ in terms of its action upon the test function ϕ. Explain why the definitions

(i) $(gf, \phi) = (f, g\phi)$, (ii) $(f^n, \phi) = (-1)^n (f, \phi^n)$, (iii) $(S_b f, \phi) = (f, S_{-b}\phi)$,

are sensible where g is a fairly good function, f is a generalised function, h^n is the nth derivative of h and $S_b h(t) = h(t-b)$.

Show that

$$\exp(at)\delta^n(t-b) = \exp(ab) \sum_{r=0}^{r=n} \binom{n}{r} (-a)^{n-r} \delta^r (t-b),$$

where $\binom{n}{r}$ denotes the binomial coefficient.