UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M241: Mathematical Methods 3

COURSE CODE	:	**MATHM241**
UNIT VALUE	:	**0.50**
DATE	:	**15-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1050-3-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. The temperature $\theta(x,t)$ in a thin insulated rod of length L, made of conducting material, evolves according to the heat equation

$$\frac{1}{\alpha^2}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2},$$

where α^2 is a constant thermal diffusivity. Initially at $t = 0$, the temperature in the rod in °C is given by

$$\theta(x,0) = T_0 \sin\left(\frac{\pi x}{L}\right).$$

Derive an expression for $\theta(x,t)$ for $t > 0$ if

(a) Both ends of the rod are held at a fixed temperature of 0°C for $t > 0$.

(b) The end at $x = 0$ is held at 0°C and a perfect insulator is attached to the end at $x = L$ for $t > 0$.

Hence show that in the case of (b), the temperature at $x = L$ for $t > 0$ is given by

$$-\sum_{n=0}^{\infty} \frac{8T_0}{\pi(2n-1)(2n+3)} \exp\left\{-\frac{(2n+1)^2\pi^2\alpha^2 t}{4L^2}\right\}.$$

2. (a) Find an extremal function $y = f(x)$ of the functional

$$I[y] = \int_0^{\pi/2} y^2 - y'^2 - 2y\sin x \; dx, \quad \text{with } y(0) = 1, \quad y(\pi/2) = 0,$$

where $y' = dy/dx$. Is $f(x)$ likely to maximize or minimize $I[y]$? Briefly explain your reasoning.

(b) Find a function $y = f(x)$ to minimize the functional

$$J[y] = \int_0^1 y^2 y'^2 \, dx, \quad \text{with} \quad y(0) = 1,\ y(1) = 2,$$

subject to the constraint that

$$K[y] = \int_0^1 y^2 \, dx = 3.$$

3. Find the general solutions of the following partial differential equations

(a)
$$\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = e^{y-2x} + e^{y+2x},$$

(b)
$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = (x-y)^2$$

(c)
$$(z-3x)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + y^2.$$

In the case of (c) you may leave the solution in implicit form.

4. (a) A rightwards travelling simple harmonic wave with speed c is described by the equation
$$z(x,t) = \text{Re}\left\{A\, e^{i(\omega t - kx)}\right\}, \qquad (*)$$
where A is a complex constant, and Re$\{\cdot\}$ denotes the real part of a complex number. Define the amplitude, wavelength and period of this wave in terms of A, k and ω. A second rightwards travelling wave is also given by $(*)$, except with the constant A replaced by B. State the definition of the phase difference between the two waves in terms of A and B.

(b) An infinite string, with density ρ and under tension T, lies along the x-axis in its undisturbed state and is free to move in the plane $y = 0$. A mass M is attached to the point $x = 0$. If the amplitude of displacements to the string is given by $z_1(x,t)$ in $x < 0$ and $z_2(x,t)$ in $x > 0$, one of the boundary conditions at $x = 0$ may be shown to be
$$M\frac{\partial^2 z_1}{\partial t^2} = T\left(\frac{\partial z_2}{\partial x} - \frac{\partial z_1}{\partial x}\right) \qquad \text{at } x = 0.$$

Write down a partial differential equation describing the evolution of the displacements $z_i(x,t)$, $(i = 1,2)$ together with any further boundary conditions satisfied at $x = 0$. Care should be taken to define any constants that are introduced.

A rightwards travelling simple harmonic wave, given by $(*)$, propagates from $x \ll 0$ to be incident upon the mass at $x = 0$. Find the amplitude of the leftwards travelling wave reflected back into the string in $x < 0$, and the amplitude of the rightwards travelling transmitted wave in $x > 0$. What is the phase difference between the incident and transmitted waves?

5. Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\zeta)\, d\zeta$$

of the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \qquad (-\infty < x < \infty, \quad t \geq 0)$$

with c a constant, when the initial conditions are

$$z(x,0) = F(x), \qquad z_t(x,0) = G(x), \qquad (-\infty < x < \infty).$$

If $F(x) = 0$ for $(-\infty < x < \infty)$ and

$$G(x) = \begin{cases} \cos\left(\frac{\pi x}{2}\right) & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

display the values of $z(x,t)$ for $t > 0$ in an (x,t) plane diagram.

Sketch graphs of the solution for $t = 1/2c$ and $t = 2/c$.

6. (a) State the definition of a critical point of a continuously differentiable two-dimensional function $z = z(x,y)$. State briefly, without proof, how critical points are classified.

(b) Determine the integral surface $z = z(x,y)$ satisfying the partial differential equation

$$3y^2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = 3y^2 \cosh x - 6,$$

and the boundary data

$$z(0,y) = y^3 - 3y.$$

(c) Show that the integral surface $z = z(x,y)$ found in (b) has four critical points located at $(\pm\cosh^{-1}\{2\}, \pm 1)$. Classify each of the four critical points.