UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)M.Sci.*

Mathematics M241: Mathematical Methods 3

COURSE CODE	:	**MATHM241**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0968-3-170

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Find the general solutions of the following partial differential equations

   (a)
$$\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \sin(x+y),$$

   (b)
$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0,$$

   (c)
$$2x^2 z\frac{\partial z}{\partial x} + \cos^2 y\frac{\partial z}{\partial y} + xz^2 = 0.$$

   In the case of (c) you may leave the solution in implicit form.

2. The functional
$$I = \int_{x_1}^{x_2} F(y, y')\, dx,$$
   with the values of $y(x_1)$ and $y(x_2)$ prescribed, is minimised by $y(x)$. Assuming that Euler's equation is satisfied, deduce that
$$F - y'\frac{\partial F}{\partial y'} = \text{constant}, \qquad \text{(Beltrami's Identity)}.$$

   A boat in a semi-infinite ocean ($0 < y < \infty$) is constrained to travel at speed λy. If the boat follows a path $y(x)$ between points $(0, 1)$ and $(2, 1)$, write down a functional expression for the travel time.

   Show that the path which minimises the travel time satisfies the relation
$$(y'^2 + 1)y^2 = k^2, \qquad (k \text{ is an unknown constant}).$$

   Hence determine the minimising path and the fastest time. How does the fastest time compare to the time taken if the boat sails in a straight line between the two points?

3. Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\zeta)\, d\zeta$$

of the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \qquad (-\infty < x < \infty, \quad t \geq 0)$$

with c a constant, when the initial conditions are

$$z(x,0) = F(x), \qquad z_t(x,0) = G(x), \qquad (-\infty < x < \infty).$$

If $G(x) = 0$ for $(-\infty < x < \infty)$ and

$$F(x) = \begin{cases} 1 - \sqrt{|x|}, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$$

display the values of $z(x,t)$ for $t > 0$ in an (x,t) plane diagram.

Sketch graphs of the solution for $t = 0$, $t = 1/2c$ and $t = 2/c$.

4. Two stretched strings, each of length L, have densities of ρ_1 and ρ_2, and each is under tension T_0. When at rest, the strings lie along the x-axis $(-L \leq x \leq L)$ and are joined together at $x = 0$. The strings are fixed at $x = \pm L$.

   Write down the partial differential equations satisfied by the displacement $z(x,t)$ in each string, if the strings perform small transverse oscillations in the plane $y = 0$. Define any constants in terms of T_0, ρ_1 and ρ_2, and state the boundary conditions satisfied at $x = 0$ and $x = \pm L$.

   Show that the periods of the normal modes of vibration of the strings are $2\pi/\omega$ where ω satisfies the equation

$$c_1 \tan\left(\frac{\omega L}{c_1}\right) + c_2 \tan\left(\frac{\omega L}{c_2}\right) = 0,$$

   with c_1 and c_2 being the wave speeds along the strings with densities ρ_1 and ρ_2 respectively.

5. (a) Determine the integral surface that satisfies the partial differential equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - 2z = 0,$$

and passes through the curve

$$y = 1, \quad z = 3x^2 - 8x - 3.$$

(b) Using Lagrange multipliers, or otherwise, find the minimum distance from the hyperbola with equation

$$3x^2 - 8xy - 3y^2 = 1,$$

to the origin.

Is there a geometrical connection with the integral surface found in (a)?

6. The *steady* temperature distribution $\theta(x, y)$ of a square sheet of metal, which has its upper and lower surfaces insulated, satisfies Laplace's equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0, \qquad 0 \le x \le 1, \;\; 0 \le y \le 1.$$

(a) Derive the general solution for θ in variables separable form.

(b) Find the temperature everywhere in the sheet if the temperature is held constant ($\theta = 0$) on three edges ($y = 0$, $y = 1$ and $x = 0$) and the fourth edge at $x = 1$ is heated so that

$$\theta(1, y) = \sin 2\pi y \, \cos \pi y.$$

(c) Using arguments involving symmetry and linearity, or otherwise, write down the steady solution if the edge at $y = 0$ is also heated so that

$$\theta(x, 0) = \sin 2\pi x \, \cos \pi x,$$

with the temperatures on the other three edges as in (b).