UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics M241: Mathematical Methods 3

COURSE CODE	:	**MATHM241**
UNIT VALUE	:	**0.50**
DATE	:	**06–MAY–04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04–C0962–3–180

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Examine the function $z = x^3 - 3x + xy^2$ for relative maxima, minima and saddle points.

   (b) Find the stationary points of the function $u = \sqrt{x^2 + y^2}$, subject to the constraint $x^2 + y^2 + 2x - 2y + 1 = 0$.

2. (a) Find the general solution to the partial differential equation

$$3x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xy^2 \ ,$$

   and verify that the solution satisfies the given equation.

   (b) Determine the integral surface $z = f(x, y)$, which satisfies the partial differential equation

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = xy \qquad (0 < y < x),$$

   and contains the curve $z = \tan x$, $y = 0$.

3. A smooth plane curve of length L is drawn through two points placed at $P = (-a, 0)$ and $Q = (a, 0)$ in the (x, y)-plane, so as to maximize the area between the curve and the chord PQ.

   Show that this curve $y = y(x)$ satisfies the differential equation

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{\lambda^2}{(\alpha - y)^2}$$

   for some constants α and λ.

   Integrate this equation to show that the curve is an arc of a circle of radius λ, and find the centre and radius of this circle in the case when $L = \pi a$.

4. Derive the wave equation governing the displacement $z(x,t)$ of a general point of a homogeneous string of density (mass per unit length) ρ, that is stretched under uniform tension T between the end-points A and B.

   You may assume that the string performs only *small amplitude* transverse vibrations in the vertical plane (z,x), with T and ρ remaining constant, and also that gravity, air resistance and any dissipation may be neglected.

5. Two thin rods AB and CD, each of length L and constant thermal diffusivity α, are maintained at constant temperatures 0°C and 10°C respectively, for $t \leq 0$. The ends B and C are welded together at time $t = 0$, so that heat can subsequently flow freely between the rods, and the other ends A and D are now maintained at constant temperatures 0°C and 20°C respectively, for $t > 0$. By using separated variables, derive the solution to the diffusion equation, and show that for $t > 0$ we have

$$\theta(L,t) = 10 - \frac{20}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}e^{-\frac{\alpha^2(2m+1)^2\pi^2 t}{4L^2}} \quad [°\mathrm{C}] ,$$

   where $\theta(L,t)$ denotes the temperature in the centre of the rod at time t.

6. The small transverse vibrations of a circular drum are governed by the two dimensional wave equation in polar coordinates r, θ

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} ,$$

   where c is the wave speed and t is time. Look for a solution $z(r,\theta,t) = R(r)\Theta(\theta)T(t)$ expressed in separated variables form and derive the ordinary differential equations satisfied by $R(r)$, $\Theta(\theta)$ and $T(t)$.

   Give the general solutions for $T(t)$ and $\Theta(\theta)$.

END OF PAPER