UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M241: Mathematical Methods 3

COURSE CODE	:	**MATHM241**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0955-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. A heavy chain of length ℓ has density ρ per unit length and hangs under gravity from two fixed points distance $2a$ apart on the same horizontal level. Write down an integral expression for the length ℓ of the chain if coordinates (x, y) are chosen with the x-axis horizontal and the origin at one end of the chain.

   Given that the potential energy V of the chain can be written as

$$V = -\rho g \int_0^{2a} y[1 + (y')^2]^{\frac{1}{2}}\, dx\,,$$

   where g is the (constant) acceleration due to gravity, and that V is a minimum when the chain hangs in stable equilibrium, deduce that $y(x)$ satisfies the differential equation

$$\left(\frac{dy}{dx}\right)^2 = [k^2(y+h)^2 - 1]\,,$$

   where h and k are constants. Hence show that the stable equilibrium shape of the chain is a catenary with equation

$$y = -h + k^{-1}\cosh\left[k(x-a)\right].$$

   Show that equations which determine the constants h and k are

$$k\ell = 2\sinh(ka)\,, \qquad kh = \cosh(ka)\,.$$

2. (a) Find the general solution of the partial differential equation

$$x\frac{\partial z}{\partial x} - 7y\frac{\partial z}{\partial y} = x^2 y\,.$$

   (b) Determine the integral surface $z = f(x, y)$ which satisfies the partial differential equation

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = xy \qquad (0 < y < x)$$

   and contains the curve $z = \cos x,\ y = 0\,.$

3. Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}[\,F(x+ct) + F(x-ct)\,] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

of the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \qquad (-\infty < x < \infty, \quad t \geq 0)$$

with c a constant, when the initial conditions are

$$z(x,0) = F(x), \qquad z_t(x,0) = G(x) \quad (-\infty < x < \infty).$$

If $F(x) = 0$ for $-\infty < x < \infty$ and

$$G(x) = \begin{cases} \cos \pi x\,, & (|x| \leq \frac{1}{2}) \\ 0\,, & (|x| > \frac{1}{2}) \end{cases}$$

display the values of $z(x,t)$ for $t > 0$ in an (x,t) plane diagram.

If $t > 1/2c$, show that there is a range of values of x over which $z(x,t)$ has a constant value.

If $t < 1/2c$, show that there is a range of values of x over which $z(x,t)$ represents a stationary wave.

4. An infinite string lies along the x-axis which is horizontal and is stretched under constant tension T_0. For $x < 0$ the density of the string is ρ_1 while for $x > 0$ the density is ρ_2, where ρ_1 and ρ_2 are constants with $\rho_2 > \rho_1$.

Write down the differential equations which are satisfied by the displacements $z_1(x,t)$ and $z_2(x,t)$ for $x < 0$ and $x > 0$ respectively, if the string performs small transverse oscillations in the horizontal plane.

What conditions must be satisfied by $z_1(x,t)$ and $z_2(x,t)$ at $x = 0$?

A simple harmonic wave $A\,e^{i(n_1 t - k_1 x)}$, with A, k_1 and n_1 real positive constants, travels along the part of the string for which $x < 0$ in the direction of increasing x. Write down the expression for the wave speed c_1 in terms of k_1 and n_1.

The wave is partly reflected and partly transmitted at $x = 0$ on account of the discontinuity in the density of the string. Determine the amplitudes of the reflected and transmitted waves in terms of A and the ratio ρ_2/ρ_1, and comment on the limiting cases when $\rho_2/\rho_1 \to 1$ and $\rho_2/\rho_1 \to \infty$.

5. Write down the steady and unsteady solutions in variables separable form of the heat equation

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial\theta}{\partial t},$$

where $\theta(x,t)$ denotes temperature and α is the (constant) thermal diffusivity.

A thin uniform rod AB of length L, has its end A maintained at $0\,^\circ C$ and its end B maintained at $50\,^\circ C$ for $t < 0$ by heat sources. At $t = 0$ the heat source is removed from the end A which is then insulated while the temperature of the end B is maintained at $50\,^\circ C$ for $t \geq 0$.

With $x = 0$ at the end A of the rod, show that the temperature $\theta(x,t)$ in $^\circ C$ at any point of the rod for $t \geq 0$ is given by

$$\theta(x,t) = 50 - \frac{400}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\left[\frac{(2n+1)\pi x}{2L}\right]\exp\left[\frac{-(2n+1)^2\alpha^2\pi^2 t}{4L^2}\right].$$

6. Laplace's equation in plane polar coordinates (r,θ) is

$$\nabla^2\Phi \equiv \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0\,.$$

Determine all *single valued* solutions of this equation in variables separable form $R(r)\Theta(\theta)$.

The functions $\Phi_0(r,\theta)$ and $\Phi_1(r,\theta)$ are solutions of

$$\nabla^2\Phi_0 = 0\,, \qquad (r \geq 1)$$

$$\nabla^2\Phi_1 = 0\,. \qquad (0 \leq r \leq 1)$$

and satisfy the boundary conditions

(a) $\Phi_0 - r\cos\theta \to 0$ as $r \to \infty$,

(b) Φ_1 is bounded at $r = 0$,

(c) $\Phi_1 + 2\Phi_0 = 4\,, \qquad (r = 1)$

(d) $\dfrac{\partial\Phi_1}{\partial r} = \dfrac{\partial\Phi_0}{\partial r}\,. \qquad (r = 1)$

Determine the functions $\Phi_0(r,\theta)$ and $\Phi_1(r,\theta)$ and verify that

$$\int_0^{2\pi}\Phi_1(1,\theta)\,d\theta = 8\pi\,.$$