UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M241: Mathematical Methods 3

COURSE CODE	:	**MATHM241**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0950-3-140

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. The functional

$$I = \int_{x_1}^{x_2} F(y, y')\, dx\,,$$

with the values of $y(x_1)$ and $y(x_2)$ prescribed, is minimised by $y(x)$. Assuming that Euler's equation is satisfied, deduce that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant}\,.$$

The functional

$$\int_{-a}^{a} y[1 + (y')^2]^{1/2}\, dx$$

is minimised subject to the constraint

$$\int_{-a}^{a} [1 + (y')^2]^{1/2}\, dx = 2\ell\,,$$

where $\ell > a$, together with the boundary conditions $y(-a) = y(a) = 0$.
Show that

$$\frac{y + \lambda}{[1 + (y')^2]^{1/2}} = c\,,$$

where c and λ are constants, and deduce that the minimising function is

$$y = c\,[\cosh(x/c) - \cosh(a/c)]\,,$$

where $\ell = c\sinh(a/c)$.
[You may assume that $\cosh A - \cosh B = 2\,\sinh\frac{1}{2}(A+B)\,\sinh\frac{1}{2}(A-B)$.]

2. (a) Find the general solution of the partial differential equation

$$\frac{\partial z}{\partial x}\sec x + \frac{\partial z}{\partial y} = \cos y\,.$$

(b) Find the solution of the partial differential equation

$$(x^2+1)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} - xz = 0$$

which is such that $z = (x^2+1)^2$ when $y = 1$ for $\frac{1}{2} \le x \le \frac{2}{3}$. Indicate in a diagram the region of the (x, y) plane over which this solution is defined.

3. Derive D'Alembert's solution

$$z(x,t) = \frac{1}{2}[\,F(x+ct) + F(x-ct)\,] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

of the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}, \qquad (-\infty < x < \infty, \;\; t \geq 0)$$

with c a constant, when the initial conditions are

$$z(x,0) = F(x), \qquad z_t(x,0) = G(x) \qquad (-\infty < x < \infty).$$

If $G(x) = 0$ for $-\infty < x < \infty$ and

$$F(x) = \begin{cases} \cos^2\left(\frac{1}{2}\pi x\right), & (|x| \leq 1) \\ 0, & (|x| > 1) \end{cases},$$

display the values of $z(x,t)$ for $t > 0$ in an (x,t) plane diagram. Deduce that $z(x, 1/2c) = \frac{1}{2}$ for $|x| \leq \frac{1}{2}$.

Sketch the graphs of the solution profiles when $t = 0, 1/2c$.

4. Laplace's equation in plane polar coordinates (r,θ) is

$$\nabla^2\Phi \equiv \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0\,.$$

Find all solutions of this equation of the form $f(r)\cos n\theta$, $(n = 0, 1, \ldots)$.

The functions $\Phi_0(r,\theta)$ and $\Phi_1(r,\theta)$ are solutions of

$$\nabla^2\Phi_0 = 0\,, \quad (r > 1) \qquad\qquad \nabla^2\Phi_1 = 0\,, \quad (0 \leq r < 1)$$

and satisfy the boundary conditions

(a) $\Phi_0 - 2r\cos\theta \to 0$ as $r \to \infty$,

(b) $r\Phi_1 \to 0$ as $r \to 0$,

(c) $\Phi_0 + \Phi_1 = 3\,, \qquad (r = 1)$

(d) $\dfrac{\partial \Phi_0}{\partial r} = 3\,\dfrac{\partial \Phi_1}{\partial r}\,. \qquad (r = 1)$

Determine the functions $\Phi_0(r,\theta)$ and $\Phi_1(r,\theta)$.

5. Two stretched strings, each of length L, have densities ρ_1 and ρ_2 and each has tension T_0. When at rest, the strings lie along the part of the x-axis for which $-L \le x \le L$ and they are joined together at $x = 0$. The strings are fixed at $x = \pm L$.

If the strings perform small transverse vibrations in the horizontal plane $y = 0$, write down the differential equation which is satisfied by the displacement $z(x, t)$ in each string, defining any constants in terms of T_0, ρ_1 and ρ_2, and state the boundary conditions to be satisfied at $x = 0$ and $x = \pm L$.

Show that the periods of the normal modes of vibration of the strings are $2\pi/\omega$, where ω satisfies the equation

$$c_1 \tan\left(\frac{\omega L}{c_1}\right) + c_2 \tan\left(\frac{\omega L}{c_2}\right) = 0,$$

with c_1 and c_2 the wave speeds along the strings with densites ρ_1 and ρ_2 respectively.

6. Obtain the steady and unsteady solutions in variables separable form of the heat equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial \theta}{\partial t},$$

where $\theta(x, t)$ denotes temperature and α is the (constant) thermal diffusivity.

The ends of a thin rod of length $2L$ are maintained at $0\,°C$ and the centre of the rod is heated to $100\,°C$ by an external heat source until a steady state temperature distribution is reached in the rod. Show that the steady state temperature $\theta_0(x)$ in the rod, as a function of distance x measured from one end of the rod, is given by

$$\theta_0(x) = \begin{cases} 100\,x/L\,, & (0 \le x \le L) \\ 100\,(2L - x)/L\,, & (L \le x \le 2L) \end{cases}.$$

Verify that

$$\int_L^{2L} \theta_0(x) \sin\left(\frac{n\pi x}{2L}\right) dx = (-1)^{n+1} \int_0^L \theta_0(x) \sin\left(\frac{n\pi x}{2L}\right) dx. \qquad (n = 1, 2, \ldots)$$

The heat source is removed from the centre of the rod at time $t = 0$ but the ends $x = 0, 2L$ are maintained at $0\,°C$ for $t \ge 0$. Find the temperature distribution $\theta(x, t)$ in the rod for $t \ge 0$, and deduce that the temperature at the centre of the rod is

$$\frac{800}{\pi^2} \sum_{m=0}^{\infty} \frac{\exp\left[-\alpha^2 (2m+1)^2 \pi^2 t / 4L^2\right]}{(2m+1)^2}.$$

[You may assume that

$$\int_0^L x \sin\left(\frac{n\pi x}{2L}\right) dx = \frac{4L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2L^2}{n\pi} \cos\left(\frac{n\pi}{2}\right). \qquad (n = 1, 2, \ldots)]$$