UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS2246**
ASSESSMENT PATTERN	:	**PHAS2246A**
MODULE NAME	:	**Mathematical Methods III**
DATE	:	**20-May-09**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

2008/09-PHAS2246A-001-EXAM-187

TURN OVER

All questions, including just parts of questions, may be attempted. Credit will be given for all correct work done.

For guidance: A student should aim to answer correctly the equivalent of five questions in the time available.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

1. (a) Define what is meant by a "conservative vector field". If the curl of a field $\underline{F}$ vanishes, then what can one say about the field $\underline{F}$ being conservative or not? [3 marks]

The vector fields $\underline{F}_1(x, y, z)$ and $\underline{F}_2(x, y, z)$ are defined, in Cartesian coordinates, by:

$$\begin{aligned} \underline{F}_1(x,y,z) &= (2xy - z^5)\hat{\underline{e}}_x + x^2\hat{\underline{e}}_y - (5xz^4 + 1)\hat{\underline{e}}_z \, , \\ \underline{F}_2(x,y,z) &= (2xy - z^5)\hat{\underline{e}}_x + x^2\hat{\underline{e}}_y + (5xz^4 + 1)\hat{\underline{e}}_z \, . \end{aligned}$$

Is either of these vector fields conservative? Show which one of these field is conservative. For this field determine (up to an additive constant) the scalar potential from which such a field arises. [6 marks]

(b) Stokes' theorem states that the line integral of a vector field $\underline{G}$ along a closed loop C is equal to the flux of the curl $\underline{\nabla} \times \underline{G}$ through the surface enclosed by C:

$$\oint_C \underline{G} \cdot \mathrm{d}\underline{r} = \int_C (\underline{\nabla} \times \underline{G}) \cdot \mathrm{d}\underline{S} \, ,$$

where $\mathrm{d}\underline{S}$ points towards the region of space from where an observer would see the loop integral as anti-clockwise.

Consider the vector field $\underline{G}(x, y, z)$ given, in Cartesian coordinates, by:

$$\underline{G}(x,y,z) = 2y\hat{\underline{e}}_x - 3x\hat{\underline{e}}_y + z^2\hat{\underline{e}}_z \, .$$

Find the curl $\underline{\nabla} \times \underline{G}$. Evaluate the line integral $\oint \underline{G}(x, y, z) \cdot \mathrm{d}\underline{r}$ around the square lying in the xy plane ($z = 0$) and bounded by the lines $x = 3$, $x = 5$, $y = 1$ and $y = 3$, either directly or by applying Stokes' theorem (take the line integral anti-clockwise as seen from the positive z semi-space). [5 marks]

(c) By virtue of the divergence theorem, the outward flux of a vector field $\underline{G}$ through any closed surface S is equal to the volume integral of its divergence $\underline{\nabla} \cdot \underline{G}$ over the volume V enclosed by the surface:

$$\int_S \underline{G} \cdot \mathrm{d}\underline{S} = \int_V \underline{\nabla} \cdot \underline{G} \, \mathrm{d}V \, ,$$

where $\mathrm{d}\underline{S}$ points outward from the closed surface.

Consider the scalar field in Cartesian coordinates:

$$H(x,y,z) = x^3 + xy^2 - z \, .$$

Express H in cylindrical polar coordinates ρ, θ and z (where $\rho^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$). Given the expression for the Laplacian in cylindrical polar coordinates

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2},$$

determine $\nabla^2 H(\rho, \theta, z)$.

Either directly or by applying the divergence theorem, evaluate the outgoing flux of the gradient $\underline{\nabla} H$, given by

$$\int \underline{\nabla} H \cdot \mathrm{d}\underline{S},$$

over the total surface of a cylinder of radius R and height h with its base lying on the $z = 0$ plane and centred at the origin. [6 marks]

2. Consider the following second-order linear differential equation

$$x\frac{d^2y}{dx^2} + (2 - x)\frac{dy}{dx} + by = 0, \tag{1}$$

where b is a constant. By writing equation (1) in the form $y'' + p(x)y' + q(x)y = 0$, or otherwise, determine where this equation is singular. [2 marks]

Solutions of equation (1) can be written in the form:

$$y = \sum_{n=0}^{\infty} a_n x^{n+k}, \qquad a_0 \neq 0.$$

Show that $k = 0$ or $k = -1$. [4 marks]

Derive the recurrence relation

$$a_{n+1} = \frac{n + k - b}{(n + k + 1)(n + k + 2)} a_n.$$ [5 marks]

Demonstrate that the series solutions converge for all values of x. [2 marks]

In the special case of $b = m$, a positive integer, show that the series with $k = 0$ terminates at $n = m$ to yield a polynomial solution. [3 marks]

Obtain this solution for the case of $b = m = 2$ and demonstrate that it satisfies the differential equation (1). [4 marks]

3. If a matrix $\underline{H}$ is described as Hermitian, what property does it have? Prove that the eigenvalues of a Hermitian matrix are real. What property must the associated eigenvectors have? **[7 marks]**

The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} 5 & -5 & 1 \\ -5 & 11 & -5 \\ 1 & -5 & 5 \end{pmatrix} .$$

Is $\underline{A}$ Hermitian? What is its trace? **[2 marks]**

Verify that $\lambda_1 = 16$ is an eigenvalue of $\underline{A}$ and that its associated eigenvector can be written as $\underline{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Show that $\lambda_2 = 4$ is also an eigenvalue and obtain the third eigenvalue, λ_3. Find the *normalised* eigenvectors corresponding to eigenvalues λ_2 and λ_3. **[11 marks]**

4. A particle of mass $m = 1$ is moving on a plane. Its position is represented, in Cartesian coordinates, by the two-dimensional vector $\underline{r} =$

The particle is subject to a conservative force field $\underline{F}(\underline{r})$, with potential energy $U(\underline{r})$

$$U(\underline{r}) = \frac{2-\sqrt{2}}{\quad}\ {}_{2} \quad \frac{1-\sqrt{2}}{\quad}\ {}_{2} \quad \frac{1}{\quad}(\qquad)^2$$

Write down the general relation between the force $\underline{F}(\underline{r})$ and the potential energy $U(\underline{r})$ $\underline{F}(\underline{r})$ acting on the particle. **[3 marks]**

Find the work done against the force $\underline{F}(\underline{r})$ when the particle moves from the point $(0, 0$ **[2 marks]**

Write down the particle's equation of motion and show that it can be written as

$$\ddot{\underline{r}} = \underline{A}\,\underline{r}$$

where $\underline{A}$ is the 2×2 real matrix **[4 marks]**

$$\underline{A} = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} .$$

Find the eigenvalues and eigenvectors of $\underline{A}$. **[5 marks]**

Use these eigenvalues and eigenvectors to obtain two uncoupled differential equations describing the motion. Solve these equations for the initial conditions $x =$ $y = \frac{dy}{dt} = 0$
y. **[6 marks]**

5. In atomic units the Schrödinger equation for the hydrogen atom can be written as

$$\left(-\frac{1}{2}\nabla^2 - \frac{1}{r}\right)\psi = E\psi$$

where, in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

By writing

$$\psi = R(r)\Theta(\theta)\Phi(\phi)$$

show that $\Phi(\phi)$ must satisfy the equation

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi\,.$$

[4 marks]

What are the solutions of this equation? Explain why m, the constant of separation, must take integer values. **[3 marks]**

Hence show that $R(r)$ must satisfy the radial equation

$$\frac{1}{2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + Rr + ERr^2 = \lambda R\,,$$

where λ is another constant of separation. Obtain the corresponding equation for $\Theta(\theta)$. **[6 marks]**

For the special case of $\lambda = 0$, show that the radial equation can be written as

$$\frac{1}{2}\frac{d^2U}{dr^2} + \left[\frac{1}{r} + E\right]U = 0$$

where $U(r) = rR(r)$. Show that for large values of r **[3 marks]**

$$U(r) = A\exp(\alpha r) + B\exp(-\alpha r)$$

where $\alpha = (-2E)^{\frac{1}{2}}$. How can this solution be simplified for bound states of the hydrogen atom? **[4 marks]**

6. (a) Any continuous function $f(x)$ in $-1 \leq x \leq 1$ can be expanded in terms of Legendre polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{for} \quad -1 \leq x \leq +1 \, .$$

Given the orthogonality relation

$$\int_{-1}^{+1} P_m(x) P_n(x) \, \mathrm{d}x = \delta_{mn} \frac{2}{2n+1} \, ,$$

derive a formula for the coefficients a_n. **[5 marks]**

Given the first two Legendre polynomials

$$P_0(x) = 1 \, , \quad P_1(x) = x \, ,$$

find the first two coefficients a_0 and a_1 of the expansion of the function $\alpha \, \mathrm{e}^{\alpha |x|}$, where α is real. **[5 marks]**

(b) The electrostatic potential of a charge Q located on the z axis at the point $(0, 0, d)$ (in Cartesian coordinates; note that d can be either positive or negative), reads, in spherical polar coordinates:

$$V(r, \theta, \qquad \frac{Q}{4\pi\varepsilon_0 r} \; {}^{\infty}_{l=0} \left(\frac{d}{r}\right)^{l}$$

Add now a second charge $-Q$ on the z axis at point $(0, 0, -d)$. Write down an expression for the total potential V_T created by the charges Q and $-Q$ in spherical polar coordinates. **[4 marks]**

Consider a charge q very far from the origin ($d \ll r$). By approximating the total potential V_T due to charges Q and $-Q$ to the first non-zero term of the expansion in Legendre polynomials, and using the expression of the gradient in spherical polars (given below), find the electrostatic force acting on the charge q in this case. **[6 marks]**

The gradient operator in spherical polar coordinates is

$$\underline{\nabla} = \hat{\underline{e}}_r \frac{\partial}{\partial r} + \hat{\underline{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\underline{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \, .$$

7. (a) Let $f(x)$ be a function of period π defined by

$$f(x) = \sin(x) \quad \text{for} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} .$$

Is $f(x)$ an even or odd function?

Show that the Fourier expansion of $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(2nx)$$

where the coefficients b_n are given by **[4 marks]**

$$b_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \sin(2nx) \sin(x) \, \mathrm{d}x .$$

Determine the coefficients b_n and hence show that **[5 marks]**

$$f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{8n}{1-4n^2} \sin(2nx) .$$

Parseval's identity for a function $f(x)$ with general period $2L$ reads

$$\frac{1}{2L} \int_{-L}^{+L} [f(x)]^2 \, \mathrm{d}x = (a_0/2)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) .$$

Apply Parseval's identity to prove **[3 marks]**

$$\sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2} = \frac{\pi^2}{64} .$$

(b) The function $g(x)$ is defined by

$$\begin{aligned} g(x) &= \sin(x) \quad \text{for} \quad -l < x < +l , \\ g(x) &= 0 \quad \text{for} \quad |x| \geq l , \end{aligned}$$

where l is real and positive. The Fourier transform $\tilde{g}(k)$ of $g(x)$ is defined as

$$\tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-ikx} g(x) \, \mathrm{d}x .$$

Give the general way to obtain the original function $g(x)$ from its Fourier transform $\tilde{g}(k)$ (*i.e.*, to obtain the inverse Fourier transform of $\tilde{g}(k)$). **[3 marks]**

Prove that, in the limit $l \to +\infty$, one has **[5 marks]**

$$\lim_{l \to \infty} \tilde{g}(k) = i \sqrt{\frac{\pi}{2}} \left[\delta(k+1) - \delta(k-1) \right] ,$$

where δ stands for the Dirac delta function.

END OF PAPER

UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **PHAS2246**

ASSESSMENT PATTERN : **PHAS2246A**

MODULE NAME : **Mathematical Methods III**

DATE : **30-Apr-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 30 Minutes**

2009/10-PHAS2246A-001-EXAM-139

TURN OVER

Answer ALL SIX questions in Section A and THREE questions from Section B.

The provisional allocation of maximum marks per sub-section of a question is indicated in brackets in the right-hand margin.

Section A

1. Give a definition of a *conservative* vector field and write down the relationship between a conservative vector field **F** and its potential V. [3 marks]

   Given a generic vector field in Cartesian coordinates $\mathbf{G}(x, y, z)$, write an expression for its curl $\nabla \times \mathbf{G}$ in Cartesian components. [1 mark]

   Evaluate the curl of a conservative vector field **F**. [1 mark]

2. Define the eigenvector **v** and associated eigenvalue λ of a matrix M. [2 marks]

   Define the transpose M^T of the matrix M with entries M_{jk}. [1 mark]

   Give the definition of a *symmetric* matrix S. [1 mark]

   Define the Hermitian conjugate $H^\dagger$ of the matrix H with complex entries H_{jk}. [1 mark]

   Give the definition of a *Hermitian* matrix. [1 mark]

   Show that the eigenvalues of a Hermitian matrix are real. [3 marks]

3. Consider the partial differential equation

$$\frac{\partial^2}{\partial x^2} f = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f \, .$$

   Assume $f(x, t) = X(x)T(t)$ and separate the equation into two ordinary differential equations for X and T. [4 marks]

   Solve the equations for $X(x)$ and $T(t)$, and hence give the general solution for the partial differential equation. [3 marks]

4. Consider the differential equation

$$(x^2 - 1)y'' + \frac{1}{x} y' - by = 0 \, ,$$

   where b is a constant. Re-write the equation in the form: [1 mark]

$$y'' + p(x)y' + q(x)y = 0 \, .$$

   Find all the singular points of the equation and determine whether they are regular or essential singularities. [2 marks]

Hence, explain why a series solution of the form

$$y = \sum_{j=0}^{\infty} a_j x^{j+k}$$

exists. [2 marks]

Without solving the equation, evaluate the radius of convergence of such a solution. [1 mark]

5. The Legendre polynomials $P_l(\mu)$ satisfy Legendre's differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}P_l(\mu)}{\mathrm{d}\mu}\right] + l(l+1)P_l(\mu) = 0\,.$$

Prove the orthogonality relation

$$\int_{-1}^{+1} P_m(\mu)P_n(\mu)\mathrm{d}\mu = 0 \quad \text{for} \quad m \neq n$$

by following the steps below:

- write down Legendre's equation for $l = m$ and $l = n$;
- multiply the equation for $l = m$ through by $P_n(\mu)$;
- multiply the equation for $l = n$ through by $P_m(\mu)$;
- integrate both sides of both equations between -1 and $+1$; [4 marks]
- subtract one equation from the other and impose the condition $m \neq n$ to obtain the orthogonality relation. [3 marks]

6. Any continuous function $f(x)$ of period 2π can be expanded as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(nx) + b_n\sin(nx)]\,.$$

Explain why the Fourier series of an odd function $g(x)$ simplifies to [2 marks]

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,.$$

Using the relationship

$$\int_{-\pi}^{+\pi} \sin(mx)\sin(nx)\mathrm{d}x = \pi\delta_{mn}\,,$$

show that [4 marks]

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} \sin(nx)g(x)\,\mathrm{d}x\,.$$

Section B

7. Stokes' theorem states that the line integral of a vector field F along a closed path C equals the flux of the curl $\nabla \times \mathbf{F}$ through the surface enclosed by C:

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} ,$$

where dS points towards the region of space from which the line integral appears to be taken in the counter-clockwise direction.

Consider the following vector field

$$\mathbf{F} = (2x - y)\hat{\mathrm{e}}_x - z\,\hat{\mathrm{e}}_y - \gamma \mathrm{e}^{-z/\gamma}\hat{\mathrm{e}}_z .$$

where γ is a constant.

By utilising Stokes' theorem, evaluate the line integral of **F** along a closed path of area $A_1 = 1$ lying on the $z = 0$ plane (take the integral counter-clockwise if seen from the positive z semi-space). **[5 marks]**

Also, by invoking the theorem once again, evaluate the line integral of F along a closed path of area $A_2 = 3$ lying on the $y = 0$ plane (take the integral counter-clockwise if seen from the positive y semi-space). **[5 marks]**

By virtue of the divergence theorem, the outward flux of the vector field F through any closed surface S equals the integral of its divergence $\nabla \cdot \mathbf{F}$ over the volume enclosed by the surface:

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \nabla \cdot \mathbf{F}\, \mathrm{d}V ,$$

where dS points outward from the closed surface.

By applying the divergence theorem, evaluate the surface integral of **F** through the surface of a cube of side l, with a vertex at the origin and the opposite vertex at the point (l, l, l) in Cartesian coordinates. **[10 marks]**

8. Consider the matrix

$$A = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 1 & -4 \\ -1 & -4 & 1 \end{pmatrix} .$$

Is this matrix diagonalisable? Why? **[2 marks]**

Verify that $\lambda_1 = 6$ is an eigenvalue of A with associated eigenvector **[4 marks]**

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Find the two remaining eigenvalues and their corresponding *normalised* eigenvectors. **[14 marks]**

9. Consider again the differential equation discussed in Question 4:

$$(x^2 - 1)y'' + \frac{1}{x}y' - by = 0\,,$$

where b is a constant. What happens to the equation if the sign of x is flipped? What consequences does the behaviour of the equation when the sign of x is flipped have on the solutions of the equation? **[1 mark]** **[2 mark]**

As noted in Question 4, this equation admits series solutions of the form:

$$y = \sum_{j=0}^{\infty} a_j x^{j+k}\,.$$

By solving the indicial equation or otherwise, show that the allowed values of k are 0 and 2. **[4 marks]**

For $k = 2$, show that the recurrence relation is **[6 marks]**

$$a_{j+2} = \frac{(j+2)(j+1) - b}{(j+4)(j+2)} a_j\,.$$

Determine the radius of convergence of the series solution. **[3 marks]**

For $k = 2$ and $b = 12$, show that the series may terminate and that one solution reduces to a polynomial. Determine such a polynomial solution by setting $a_0 = 1$. **[4 marks]**

10. A continuous function $f(\mu)$ between $\mu = -1$ and $\mu = +1$ can be expanded as follows in terms of Legendre polynomials:

$$f(\mu) = \sum_{l=0}^{\infty} c_l P_l(\mu)\,,$$

with coefficients c_l given by

$$c_l = \frac{2l+1}{2} \int_{-1}^{+1} P_l(\mu) f(\mu) \mathrm{d}\mu\,.$$

Given the first Legendre polynomial $P_0(\mu) = 1$, evaluate the coefficient c_0 of the Legendre expansion of **[8 marks]**

$$f(\mu) = \frac{\mu - \frac{1}{2}}{\left(\frac{5}{4} - \mu\right)^{3/2}}\,.$$

Legendre polynomials obey the generating function relationship

$$\frac{1}{(1 - 2\mu t + t^2)^{1/2}} = \sum_{l=0}^{\infty} t^l P_l(\mu)\,.$$

By differentiating the generating function with respect to t, and by inserting an appropriate value of t, determine all the coefficients of the expansion of $f(\mu)$. **[12 marks]**

11. Let $f(x)$ be a function of period 2π defined as

$$
\begin{aligned}
f(x) &= \sin(2x) \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq +\frac{\pi}{2}\,, \\
f(x) &= 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi\,.
\end{aligned}
$$

Is $f(x)$ an even or odd function? **[1 mark]**

Show that the Fourier expansion of $f(x)$ reads **[11 marks]**

$$f(x) = \frac{1}{2}\sin(2x) + \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{4-(2m+1)^2}\sin\left[(2m+1)x\right]\,.$$

Use the expansion at $x = \pi/2$ to prove that **[3 marks]**

$$\sum_{m=0}^{\infty}\frac{1}{4-(2m+1)^2} = 0\,.$$

Parseval's identity for a Fourier expansion between $-\pi$ and $+\pi$ reads:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[f(x)\right]^2\,\mathrm{d}x = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)\,.$$

Apply Parseval's identity to show that **[5 marks]**

$$\sum_{m=0}^{\infty}\frac{1}{[4-(2m+1)^2]^2} = \frac{\pi^2}{64}\,.$$

END OF PAPER

UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS2246**
ASSESSMENT PATTERN	:	**PHAS2246A**
MODULE NAME	:	**Mathematical Methods III**
DATE	:	**23-May-11**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

2010/11-PHAS2246A-001-EXAM-139

TURN OVER

Answer ALL SIX questions in Section A and THREE questions from Section B.

The provisional allocation of maximum marks per sub-section of a question is indicated in brackets in the right-hand margin.

Section A

1. Give a definition of a *conservative* vector field and write down the relationship between a conservative vector field $\mathbf{F}$ and its potential V. [3 marks]

   Given a generic vector field in Cartesian coordinates $\mathbf{G}(x, y, z)$, write an expression for its divergence $\nabla \cdot \mathbf{G}$ and for its curl $\nabla \times \mathbf{G}$ in Cartesian components. [2 marks]

   Express the divergence of a conservative vector field in terms of the Laplacian $\nabla^2 V$ of its potential. [1 mark]

   Evaluate the curl of a conservative vector field $\mathbf{F}$. [1 mark]

2. Consider the following partial differential equation:

$$\kappa \frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} ,$$

   where κ is a constant. Assume $f(x,t) = X(x)T(t)$ and separate the equation into two ordinary differential equations for $X(x)$ and $T(t)$. [4 marks]

   Solve the equations for $X(x)$ and $T(t)$, and hence give the general solution for the partial differential equation. [3 marks]

3. The Laguerre differential equation reads

$$xy'' + (1-x)y' + ny = 0 ,$$

   where n is a constant. Re-write the equation in the form: [1 mark]

$$y'' + p(x)y' + q(x)y = 0 .$$

   Find all the singular points of the equation and determine whether they are regular or essential singularities. [2 marks]

   Hence, explain why a series solution of the form

$$y = \sum_{j=0}^{\infty} a_j x^{j+k}$$

   exists. [2 marks]

   Without solving the equation, evaluate the radius of convergence of such a solution. [1 mark]

4. Define the eigenvector v and associated eigenvalue λ of a matrix M. **[2 marks]**

   Describe a sufficient condition for a matrix M to be *diagonalisable*. **[2 marks]**

   Show that the eigenvalues of a Hermitian matrix are real. **[3 marks]**

5. Any continuous function $f(\mu)$ can be expanded, over the interval $[-1,+1]$, as a series of Legendre polynomials:

$$f(\mu) = \sum_{l=0}^{\infty} c_l P_l(\mu)\,, \quad \mu \in [-1,+1]\,.$$

   Using the relationship

$$\int_{-1}^{+1} P_l(\mu) P_m(\mu) \mathrm{d}\mu = \frac{2}{2l+1}\delta_{lm}\,,$$

   show that **[3 marks]**

$$c_l = \frac{2l+1}{2}\int_{-1}^{+1} P_l(\mu) f(\mu) \mathrm{d}\mu\,.$$

   Then, given the first three Legendre polynomials

$$P_0(\mu) = 1\,, \quad P_1(\mu) = \mu\,, \quad P_2(\mu) = (3\mu^2 - 1)/2\,,$$

   determine the coefficients of the expansion of the function **[3 marks]**

$$f(\mu) = \mu^2\,.$$

6. Any continuous function $f(x)$ of period $2L$ can be expanded as a Fourier series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L)]\,.$$

   Consider the function $f(x) = \sin(x)\cos(x)$.

   What is the period of $f(x)$? **[2 marks]**

   Is $f(x)$ even or odd? What consequences does the parity of $f(x)$ have on the coefficients of its Fourier series? **[2 marks]**

   Determine all the coefficients of the Fourier series of $f(x)$. **[3 marks]**

Section B

7. Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 4 \\ -1 & 1 & -4 \\ 4 & -4 & -2 \end{pmatrix} .$$

Is this matrix diagonalisable? Why? **[2 marks]**

Verify that v_1, given below, is an eigenvector of A and determine its associated eigenvalue: **[6 marks]**

$$\mathrm{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} .$$

Find the two remaining eigenvalues and their corresponding *normalised* eigenvectors. **[12 marks]**

8. The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined as

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-ikx} f(x) \mathrm{d}x .$$

A function and its transform are also related by Parseval's identity:

$$\int_{-\infty}^{+\infty} |f(x)|^2 \mathrm{d}x = \int_{-\infty}^{+\infty} |\tilde{f}(k)|^2 \mathrm{d}k .$$

Determine the Fourier transform of the function **[10 marks]**

$$f(x) = \mathrm{e}^{-|x|} .$$

Hence, apply Parseval's identity to determine the value of the integral **[6 marks]**

$$\int_{-\infty}^{+\infty} \frac{1}{(1+k^2)^2} \mathrm{d}k .$$

Also, determine the Fourier transform of the function **[4 marks]**

$$g(x) = \mathrm{e}^{+i\vartheta x - |x|} .$$

9. The Legendre polynomials obey the following generating function relationship:

$$g(\mu,t)=\frac{1}{\sqrt{1-2\mu t+t^2}}=\sum_{l=0}^{\infty}P_l(\mu)t^l\ .$$

Show that **[6 marks]**

$$(\mu-t)\frac{\partial g(\mu,t)}{\partial\mu}=t\frac{\partial g(\mu,t)}{\partial t}\ .$$

Next, by inserting the expansion of $g(\mu,t)$ in terms of Legendre polynomials in the previous formula, obtain the following recursion relation (where $P_l'(\mu)$ stands for $\frac{\mathrm{d}P_l}{\mathrm{d}\mu}$): **[10 marks]**

$$lP_l(\mu)=\mu P_l'(\mu)-P_{l-1}'(\mu)\,,\quad \text{for}\quad l\geq 1\,.$$

Check such a relation for $l=1$ and $l=2$, using the expression of the first three Legendre polynomials given here: **[4 marks]**

$$P_0(\mu)=1\,,\quad P_1(\mu)=\mu\,,\quad P_2(\mu)=\frac{1}{2}(3\mu^2-1)\,.$$

10. Consider the Laguerre differential equation:

$$xy''+(1-x)y'+ny=0\,,$$

where n is a constant. Does this equation have a well defined parity? **[2 marks]**

This equation admits series solutions of the form:

$$y=\sum_{j=0}^{\infty}a_j x^{j+k}\ .$$

By solving the indicial equation or otherwise, show that $k=0$. **[4 marks]**

Hence, show that the recurrence relation for this equation is **[7 marks]**

$$a_{j+1}=\frac{j-n}{(j+1)^2}a_j\ .$$

Determine the radius of convergence of the series solution. **[2 marks]**

Determine the condition on n such that the series terminates and the solution reduces to a polynomial. **[2 marks]**

By setting $a_0=1$, determine such polynomial solutions ("Laguerre polynomials") for $n=0,1,2$. **[3 marks]**

11. Stokes' theorem states that the line integral of a vector field $\mathbf{F}$ along a closed path C equals the flux of the curl $\nabla \times \mathbf{F}$ through the surface enclosed by C:

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} \,,$$

where $\mathrm{d}\mathbf{S}$ points towards the region of space from which the line integral appears to be taken in the anti-clockwise direction.

Consider the following vector field in cylindrical polar components

$$\mathbf{F} = \frac{z}{\varrho}(1 - \mathrm{e}^{-\varrho/\gamma})\,\hat{\mathrm{e}}_\varrho + \frac{z}{\varrho}\,\hat{\mathrm{e}}_\varphi + \varphi\,\hat{\mathrm{e}}_z \,,$$

where γ is a constant.

Is the field $\mathbf{F}$ conservative? **[2 marks]**

By utilising Stokes' theorem, evaluate the line integral of $\mathbf{F}$ along a closed path of area $A = 2$ lying on the $\varrho = 1$ cylindrical surface (take the integral anti-clockwise as seen from outside the cylinder). **[9 marks]**

By virtue of the divergence theorem, the outward flux of the vector field $\mathbf{F}$ through any closed surface S equals the integral of its divergence $\nabla \cdot \mathbf{F}$ over the volume enclosed by the surface:

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \nabla \cdot \mathbf{F}\,\mathrm{d}V \,,$$

where $\mathrm{d}\mathbf{S}$ points outward from the closed surface.

By applying the divergence theorem, evaluate the outward flux of $\mathbf{F}$ through the surface of a cylinder of height h, radius r, main axis along the z axis, and basis lying on the $z = 0$ plane. **[9 marks]**

The divergence and curl operators of the vector field $\mathbf{v} = v_\varrho \hat{\mathrm{e}}_\varrho + v_\varphi \hat{\mathrm{e}}_\varphi + v_z \hat{\mathrm{e}}_z$ in cylindrical polar coordinates are given by:

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \frac{1}{\varrho}\partial_\varrho(\varrho v_\varrho) + \frac{1}{\varrho}\partial_\varphi v_\varphi + \partial_z v_z \,, \\
\nabla \times \mathbf{v} &= \left(\frac{1}{\varrho}\partial_\varphi v_z - \partial_z v_\varphi\right)\hat{\mathrm{e}}_\varrho + \Big(\partial_z v_\varrho - \partial_\varrho v_z\Big)\hat{\mathrm{e}}_\varphi + \frac{1}{\varrho}\Big(\partial_\varrho(\varrho v_\varphi) - \partial_\varphi v_\varrho\Big)\hat{\mathrm{e}}_z \,.
\end{aligned}$$

END OF PAPER

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EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS2246**
ASSESSMENT PATTERN	:	**PHAS2246A**
MODULE NAME	:	**Mathematical Methods III**
DATE	:	**18-May-12**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

2011/12-PHAS2246A-001-EXAM-181

TURN OVER

Answer ALL SIX questions in Section A and THREE questions from Section B.

The provisional allocation of maximum marks per sub-section of a question is indicated in brackets in the right-hand margin.

Section A

1. State the divergence theorem by expressing the surface integral of the vector field $\mathbf{v}$ across the closed surface C in terms of a volume integral. **[3 marks]**

   State Stokes' theorem by expressing the line integral of the vector field $\mathbf{v}$ along a closed path P in terms of a surface integral. **[3 marks]**

   Given a generic vector field in Cartesian coordinates $\mathbf{v}(x, y, z)$, write an expression for its divergence $\nabla \cdot \mathbf{v}$ and for its curl $\nabla \times \mathbf{v}$ in Cartesian components. **[2 marks]**

2. Assume $u(x, y) = X(x)Y(y)$ and solve the following partial differential equation by separation of variables: **[5 marks]**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \, .$$

   From the general solution, determine the set of solutions satisfying the following boundary conditions: **[4 marks]**

$$\begin{aligned} u(0, y) &= 0 \, , \\ u(L, y) &= 0 \, . \end{aligned}$$

3. Consider the differential equation

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0 \, ,$$

   where n is a constant. Re-write the equation in the form: **[1 mark]**

$$y'' + p(x)y' + q(x)y = 0 \, .$$

   Find all the singular points of the equation and determine whether they are regular or essential singularities. **[2 marks]**

   Hence, explain why two linearly independent, converging series solutions of the form

$$y = \sum_{j=0}^{\infty} a_j x^{j+k}$$

   exist. **[1 mark]**

   Without solving the equation, evaluate the radius of convergence of such solutions. **[1 mark]**

4. Give a definition of a *hermitian* matrix. If a matrix H is hermitian, can it be diagonalised? **[2 marks]**

Give a definition of a *normal* matrix. If a matrix N is normal, can it be diagonalised? **[1 mark]**

Consider the matrix

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Can the matrix J be diagonalised? **[2 marks]**

5. Any continuous function $f(x)$ in $-1 \leq x \leq 1$ can be expanded in terms of Legendre polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{for} \quad -1 \leq x \leq +1 .$$

Given the orthogonality relation

$$\int_{-1}^{+1} P_m(x) P_n(x) \, \mathrm{d}x = \delta_{mn} \frac{2}{2n+1} ,$$

derive a formula for the coefficients a_n. **[5 marks]**

6. Prove that the set of functions $\{\mathrm{e}^{in\frac{\pi}{L}x}, \forall n \in \mathbb{Z}\}$ forms an orthonormal set with respect to the inner product **[3 marks]**

$$(f(x), g(x)) = \int_{-\pi}^{+\pi} f^*(x) g(x) \, \mathrm{d}x .$$

Hence, show that, if a function $f(x)$ of period $2L$ can be expanded as a complex Fourier series according to:

$$f(x) = \sum_{n=1}^{\infty} c_n \mathrm{e}^{in\frac{\pi}{L}x} ,$$

then the coefficients c_n are given by **[3 marks]**

$$c_n = \frac{1}{2L} \int_{-L}^{+L} \mathrm{e}^{-in\frac{\pi}{L}x} f(x) \, \mathrm{d}x .$$

Solve the integral **[2 marks]**

$$\int_{-\infty}^{+\infty} \delta(x-3) x^2 \, \mathrm{d}x ,$$

where $\delta(x-3)$ is a Dirac delta function centred at 3.

Section B

7. Prove the relationship: [7 marks]

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \,,$$

for a generic smooth vector field $\mathbf{A}$.

Next, given Maxwell's equations in the vacuum:

$$\nabla \cdot \mathbf{E} = 0 \,, \qquad \nabla \times \mathbf{E} = -\frac{1}{c^2}\partial_t \mathbf{B} \,,$$
$$\nabla \cdot \mathbf{B} = 0 \,, \qquad \nabla \times \mathbf{B} = \partial_t \mathbf{E} \,,$$

prove the following wave equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$: [6 marks]

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\partial^2_{tt}\mathbf{E} \,,$$
$$\nabla^2 \mathbf{B} = \frac{1}{c^2}\partial^2_{tt}\mathbf{B} \,.$$

The propagation in vacuum of the component E_z of an electric field along the x direction is described by the previous equation with $\partial_y E_z = \partial_z E_z = 0$ (adopting Cartesian coordinates).

Such a wave equation depends only on the variables x and t. Write it down and find its general solution by separating the variables x and t. [7 marks]

8. Consider the differential equation:

$$(1 - x^2)y'' - 3xy' + n(n+2)y = 0 \,,$$

where n is a constant. Does this equation have a well defined parity? [2 marks]

This equation admits series solutions of the form:

$$y = \sum_{j=0}^{\infty} a_j x^{j+k} \,.$$

Solving the indicial equation or otherwise, show that $k = 0$ or $k = 1$. [3 marks]

Set $k = 0$ and show that the recurrence relation for this equation is [7 marks]

$$a_{j+2} = \frac{j(j+2) - n(n+2)}{(j+2)(j+1)} a_j \,.$$

Explain why you do not need to consider the case $k = 1$. [1 marks]

Determine the radius of convergence of the series solution. [2 marks]

Determine the condition on n such that the series terminates and the solution reduces to a polynomial. [2 marks]

Determine such polynomial solutions ("Jacobi polynomials") for $n = 0, 1, 2$, up to a multiplicative factor. [3 marks]

9. Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Why is it not possible to determine whether M is diagonalisable without attempting to determine a basis of eigenvectors? **[2 marks]**

Find the eigenvalues of M. **[8 marks]**

If possible, determine a basis of eigenvectors of M and write down a matrix L which diagonalises M. Is the matrix L orthogonal? **[10 marks]**

10. A piecewise continuous function $f(x)$ of period $2L$ can be expanded as a complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{in\frac{\pi}{L}x}.$$

The function $f(x)$ and the coefficients c_n are also related by Parseval's identity:

$$\frac{1}{2L}\int_{-L}^{+L} |f(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |c_n|^2.$$

Consider the function $f(x)$ of period 2π defined as:

$$f(x) = e^{-|x|} \quad \text{for} \quad -\pi \le x \le +\pi.$$

Determine the coefficients c_n of the complex Fourier series of $f(x)$. **[12 marks]**

From the expansion of $f(x)$ as a complex Fourier series, show that **[3 marks]**

$$\pi = \sum_{n=-\infty}^{+\infty} \frac{1 - e^{-\pi}(-1)^n}{1+n^2}.$$

Apply Parseval's identity to determine the value of the infinite sum **[5 marks]**

$$\sum_{n=-\infty}^{+\infty} \left(\frac{1 - e^{-\pi}(-1)^n}{1+n^2}\right)^2.$$

11. The electrostatic potential of a charge Q located on the z axis at the point $(0, 0, d)$ (in Cartesian coordinates; note that d can be either positive or negative) reads, in spherical polar coordinates:

$$V(r, \theta, \varphi) = \frac{Q}{4\pi\varepsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{d}{r}\right)^l P_l(\cos\theta) \, .$$

(a) Consider the following distribution of charges:

$$\begin{array}{rcl} Q & \text{at} & (0,0,d)\,, \\ -Q & \text{at} & (0,0,-d)\,, \end{array}$$

where the positions are given in Cartesian coordinates.

Write down an expression for the total potential V_d created by the two charges in spherical polar coordinates. **[4 marks]**

What is the total flux across a sphere centred at the origin and with radius $R > d$ of the electric field generated by this distribution of charges? **[2 marks]**

Consider a charge q very far from the origin ($d \ll r$). By approximating the total potential V_d to the first non-zero term of the expansion in Legendre polynomials (given below), and using the expression of the gradient in spherical polars (given below), find the electrostatic force acting on the charge q. **[4 marks]**

(b) Consider the following distribution of charges:

$$\begin{array}{rcl} Q & \text{at} & (0,0,d)\,, \\ -2Q & \text{at} & (0,0,0)\,, \\ Q & \text{at} & (0,0,-d)\,, \end{array}$$

where the positions are given in Cartesian coordinates.

Write down an expression for the total potential V_q created by the three charges in spherical polar coordinates. **[4 marks]**

What is the total flux across a sphere centred at the origin and with radius $R > d$ of the electric field generated by this distribution of charges? **[2 marks]**

Consider a charge q very far from the origin ($d \ll r$). By approximating the total potential V_q to the first non-zero term of the expansion in Legendre polynomials (given below), and using the expression of the gradient in spherical polars (given below), find the electrostatic force acting on the charge q in this case. **[4 marks]**

The gradient operator in spherical polar coordinates is

$$\nabla = \hat{\mathrm{e}}_r \frac{\partial}{\partial r} + \hat{\mathrm{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\mathrm{e}}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \, .$$

First Legendre polynomials: $P_1(\mu) = \mu$, $P_2(\mu) = \frac{1}{2}(3\mu^2 - 1)$.

END OF PAPER

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EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS2246**
ASSESSMENT PATTERN	:	**PHAS2246A**
MODULE NAME	:	**Mathematical Methods III**
DATE	:	**08-May-13**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

2012/13-PHAS2246A-001-EXAM-189

TURN OVER

Answer ALL SIX questions in Section A and THREE questions from Section B.

The provisional allocation of maximum marks per sub-section of a question is indicated in brackets in the right-hand margin.

Section A

1. Consider the partial differential equation
$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.$$
Assume $f(x,t) = X(x)T(t)$ and separate the equation into two ordinary differential equations for X and T. **[4 marks]**

Solve the equations for $X(x)$ and $T(t)$, and hence give the general solution for the partial differential equation. **[4 marks]**

2. Consider the following second-order linear differential equation
$$x\frac{d^2y}{dx^2} + (2-x)\frac{dy}{dx} + by = 0\,, \qquad (1)$$
where b is a constant. By writing equation (1) in the form $y'' + p(x)y' + q(x)y = 0$, or otherwise, determine whether this equation is singular at $x = 0$. **[2 marks]**

Find $p_0 = \lim_{x\to 0} xp(x)$ and $q_0 = \lim_{x\to 0} x^2 q(x)$ and hence determine whether the singularity is a regular singularity or essential singularity. **[2 marks]**

Solutions of equation (1) can be written in the form:
$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \quad a_0 \neq 0.$$
Show that $k = 0$ or $k = -1$. **[2 marks]**

3. Using
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}\,,$$
for a generic smooth vector field $\mathbf{A}$ and given the Maxwell's equations in the vacuum:
$$\begin{aligned}\nabla\cdot\mathbf{E} &= 0\,,\\ \nabla\cdot\mathbf{B} &= 0\,,\\ \nabla\times\mathbf{E} &= -\partial_t\mathbf{B}\,,\\ \nabla\times\mathbf{B} &= \frac{1}{c^2}\partial_t\mathbf{E}\,,\end{aligned}$$
prove the following wave equations for the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$: **[6 marks]**
$$\begin{aligned}\nabla^2\mathbf{E} &= \frac{1}{c^2}\partial^2_{tt}\mathbf{E}\,,\\ \nabla^2\mathbf{B} &= \frac{1}{c^2}\partial^2_{tt}\mathbf{B}\,.\end{aligned}$$

4. If a matrix $\underline{H}$ is described as Hermitian, what property does it have? Prove that the eigenvalues of a Hermitian matrix are real. What property must the associated eigenvectors have? **[8 marks]**

5. Any continuous function $f(\mu)$ can be expanded, over the interval $[-1,+1]$, as a series of Legendre polynomials:

$$f(\mu) = \sum_{l=0}^{\infty} c_l P_l(\mu) \, , \quad \mu \in [-1,+1] \, .$$

Using the relationship

$$\int_{-1}^{+1} P_l(\mu) P_m(\mu) \mathrm{d}\mu = \frac{2}{2l+1} \delta_{lm} \, ,$$

show that **[3 marks]**

$$c_l = \frac{2l+1}{2} \int_{-1}^{+1} P_l(\mu) f(\mu) \mathrm{d}\mu \, .$$

Then, given the first three Legendre polynomials

$$P_0(\mu) = 1 \, , \quad P_1(\mu) = \mu \, , \quad P_2(\mu) = (3\mu^2 - 1)/2 \, ,$$

determine the coefficients of the expansion of the function **[3 marks]**

$$f(\mu) = \mu^2 - 1 \, .$$

6. The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is defined by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

Using the above show that the Fourier transforms of $f'(x)$ and $f(x+a)$ are $ik\tilde{f}(k)$ and $e^{ika}\tilde{f}(k)$. **[6 marks]**

Section B

7. Consider the matrix

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

Find the eigenvalues of M. **[6 marks]**

Determine the eigenvectors of M and also normalize these eigenvectors. **[9 marks]**

Write down the transformation L needed to convert the matrix to a diagonal form D and show by explicit multiplication $L^{-1}ML = D$. **[5 marks]**

8. Let $f(x)$ be a function of period 2π defined as

$$\begin{aligned} f(x) &= 1 \quad \text{for} \quad -\frac{\pi}{2} \leq x \leq +\frac{\pi}{2} , \\ f(x) &= 0 \quad \text{for} \quad \frac{\pi}{2} < |x| < \pi . \end{aligned}$$

Is $f(x)$ an even or odd function? **[1 mark]**

Show that the Fourier expansion $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$ of $f(x)$ reads **[10 marks]**

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=\text{odd}} \frac{(-1)^{\frac{(n-1)}{2}}}{n} \cos(nx) .$$

Use the expansion at $x = 0$ to prove that **[4 marks]**

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4} .$$

Parseval's identity for a Fourier expansion between $-\pi$ and $+\pi$ reads:

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} [f(x)]^2 \, \mathrm{d}x = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) .$$

Apply Parseval's identity to show that **[5 marks]**

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8} .$$

9. In atomic units the Schrödinger equation for the hydrogen atom can be written as

$$\left(-\frac{1}{2}\nabla^2 - \frac{1}{r}\right)\psi = E\psi$$

where, in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

By writing

$$\psi = R(r)\Theta(\theta)\Phi(\phi)$$

separate the equation to a sum of terms with one term only a function of ϕ and the rest of the terms only a function of r and θ. **[4 marks]**

Justify why you can set the part which is only a function of ϕ to a constant. **[1 marks]**

Assume the above constant to be $-m^2$ and show that $\Phi(\phi)$ must satisfy the equation

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi\,.$$

[1 marks]

What are the solutions of this equation? Explain why m, the constant of separation, must take integer values. **[4 marks]**

Hence show that $R(r)$ must satisfy the radial equation

$$\frac{1}{2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + Rr + ERr^2 = \lambda R\,,$$

where λ is another constant. Obtain the corresponding equation for $\Theta(\theta)$. **[7 marks]**

For the special case of $\lambda = 0$, show that the radial equation can be written as

$$\frac{1}{2}\frac{d^2U}{dr^2} + \left[\frac{1}{r} + E\right]U = 0$$

where $U(r) = rR(r)$. **[3 marks]**

10. Consider the differential equation:

$$y'' - y' - 2y = 0 \, ,$$

This equation admits series solutions of the form:

$$y = \sum_{j=0}^{\infty} a_j x^{j+k} \, .$$

Construct and solve the indicial equation to show that $k = 0$ or $k = 1$. **[4 marks]**

Set $k = 0$ and show that the recurrence relation for this equation which relates a_{j+2} to a_{j+1} and a_j is

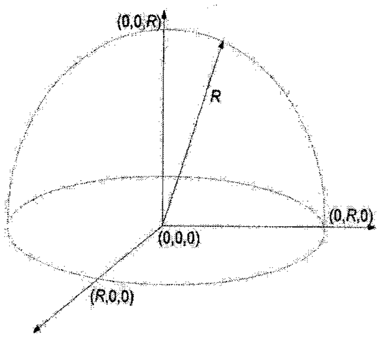
$$a_{j+2} = \frac{1}{j+2} a_{j+1} + \frac{2}{(j+1)(j+2)} a_j \, .$$

[7 marks]

Write down a_2, a_3 and a_4 in terms of a_0 and a_1 . **[3 marks]**

The general solution to the above differential equation is known to be $y = Ae^{-x} + Be^{2x}$. By comparing the first few terms of the series solution $y = \sum_{j=0}^{\infty} a_j x^{j+k}$ with the series expansion of $y = Ae^{-x} + Be^{2x}$ find A and B in terms of a_0 and a_1. [Series for $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$] **[5 marks]**

Why do we not need to find the series solution for the $k = 1$ case? **[1 marks]**



11. A vector field $\mathbf{v}$ is given by

$$\mathbf{v} = \alpha(x\hat{e}_x + y\hat{e}_y + z\hat{e}_z)$$

Find $\nabla.\mathbf{v}$ and $\nabla \times \mathbf{v}$. **[4 marks]**

Is $\mathbf{v}$ a conservative field? **[1 marks]**

Assume a *hemispherical* surface of radius R with the centre at the origin of coordinates (as shown in the above figure) and compute the flux of $\mathbf{v}$ through that surface. **[4 marks]**

Do you think there are any sources or sinks of $\mathbf{v}$ for any nonzero values of x, y and z and if so, then justify whether it is a source or a sink. **[2 marks]**

Compute the volume integral of $\nabla.\mathbf{v}$ over the hemisphere bounded by the above hemispherical surface and thereby demonstrate that the Gauss divergence theorem holds for $\mathbf{v}$. **[4 marks]**

Compute the line integral of $\mathbf{v}$ along three segments: line A connecting $(0,0,0)$ to $(0,0,R)$, arc B of radius R connecting $(0,0,R)$ to $(0,R,0)$ (this is a curved segment) and line C connecting $(0,R,0)$ to $(0,0,0)$. From the above, compute the line integral along the closed loop formed by the three segments. Could you have predicted the result of this integral along the closed loop without explicit integration and if so, why? **[5 marks]**