1. (a) Define what is meant by a "conservative vector field". If the curl of a field $\underline{F}$ vanishes, then what can one say about the field $\underline{F}$ being conservative or not? **[3]**

The vector fields $\underline{F}_1(x, y, z)$ and $\underline{F}_2(x, y, z)$ are defined, in Cartesian coordinates, by:

$$\begin{aligned} \underline{F}_1(x, y, z) &= (2xy - z^5)\hat{\underline{e}}_x + x^2\hat{\underline{e}}_y - (5xz^4 + 1)\hat{\underline{e}}_z \, , \\ \underline{F}_2(x, y, z) &= (2xy - z^5)\hat{\underline{e}}_x + x^2\hat{\underline{e}}_y + (5xz^4 + 1)\hat{\underline{e}}_z \, . \end{aligned}$$

Is either of these vector fields conservative? Show which one of these field is conservative. For this field determine (up to an additive constant) the scalar potential from which such a field arises. **[6]**

(b) Stokes' theorem states that the line integral of a vector field $\underline{G}$ along a closed loop C is equal to the flux of the curl $\underline{\nabla} \times \underline{G}$ through the surface enclosed by C:

$$\oint_C \underline{G} \cdot \mathrm{d}\underline{r} = \int_C (\underline{\nabla} \times \underline{G}) \cdot \mathrm{d}\underline{S} \, ,$$

where $\mathrm{d}\underline{S}$ points towards the region of space from where an observer would see the loop integral as anti-clockwise.

Consider the vector field $\underline{G}(x, y, z)$ given, in Cartesian coordinates, by:

$$\underline{G}(x, y, z) = 2y\hat{\underline{e}}_x - 3x\hat{\underline{e}}_y + z^2\hat{\underline{e}}_z \, .$$

Find the curl $\underline{\nabla} \times \underline{G}$. Evaluate the line integral $\oint \underline{G}(x, y, z) \cdot \mathrm{d}\underline{r}$ around the square lying in the xy plane ($z = 0$) and bounded by the lines $x = 3$, $x = 5$, $y = 1$ and $y = 3$, either directly or by applying Stokes' theorem (take the line integral anti-clockwise as seen from the positive z semi-space). **[5]**

(c) By virtue of the divergence theorem, the outward flux of a vector field $\underline{G}$ through any closed surface S is equal to the volume integral of its divergence $\underline{\nabla} \cdot \underline{G}$ over the volume V enclosed by the surface:

$$\int_S \underline{G} \cdot \mathrm{d}\underline{S} = \int_V \underline{\nabla} \cdot \underline{G} \, \mathrm{d}V \, ,$$

where $\mathrm{d}\underline{S}$ points outward from the closed surface.

Consider the scalar field in Cartesian coordinates:

$$H(x, y, z) = x^3 + xy^2 - z \, .$$

QUESTION CONTINUED

Express H in cylindrical polar coordinates ρ, θ and z (where $\rho^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$). Given the expression for the Laplacian in cylindrical polar coordinates

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \, ,$$

determine $\nabla^2 H(\rho, \theta, z)$.

Either directly or by applying the divergence theorem, evaluate the outgoing flux of the gradient $\underline{\nabla} H$, given by

$$\int \underline{\nabla} H \cdot \mathrm{d}\underline{S} \, ,$$

over the total surface of a cylinder of radius R and height h with its base lying on the $z = 0$ plane and centred at the origin. **[6]**

2. Consider the following second-order linear differential equation

$$x\frac{d^2y}{dx^2} + (2 - x)\frac{dy}{dx} + by = 0 \, , \qquad (1)$$

where b is a constant. By writing equation (1) in the form $y'' + p(x)y' + q(x)y = 0$, or otherwise, determine where this equation is singular. **[2]**

Solutions of equation (1) can be written in the form:

$$y = \sum_{n=0}^{\infty} a_n \, x^{n+k} \, , \qquad a_0 \neq 0.$$

Show that $k = 0$ or $k = -1$. **[4]**

Derive the recurrence relation

$$a_{n+1} = \frac{n + k - b}{(n + k + 1)(n + k + 2)} a_n \, .$$

Demonstrate that the series solutions converge for all values of x. **[5]**

In the special case of $b = m$, a positive integer, show that the series with $k = 0$ **[2]**
terminates at $n = m$ to yield a polynomial solution. **[3]**

Obtain this solution for the case of $b = m = 2$ and demonstrate that it satisfies the differential equation (1). **[4]**

CONTINUE

3. If a matrix $\underline{H}$ is described as Hermitian, what property does it have? Prove that the eigenvalues of a Hermitian matrix are real. What property must the associated eigenvectors have? **[7]**

The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} 5 & -5 & 1 \\ -5 & 11 & -5 \\ 1 & -5 & 5 \end{pmatrix} .$$

Is $\underline{A}$ Hermitian? What is its trace? **[2]**

Verify that $\lambda_1 = 16$ is an eigenvalue of $\underline{A}$ and that its associated eigenvector can be written as $\underline{v}_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Show that $\lambda_2 = 4$ is also an eigenvalue and obtain the third eigenvalue, λ_3. Find the *normalised* eigenvectors corresponding to eigenvalues λ_2 and λ_3. **[11]**

4. A particle of mass $m = 1$ is moving on a plane. Its position is represented, in Cartesian coordinates, by the two-dimensional vector $\underline{r} = x\underline{\hat{e}}_x + y\underline{\hat{e}}_y$.

The particle is subject to a conservative force field $\underline{F}(\underline{r})$, with potential energy $U(\underline{r})$ given by

$$U(\underline{r}) = \frac{2-\sqrt{2}}{2}x^2 + \frac{1-\sqrt{2}}{2}y^2 + \frac{1}{\sqrt{2}}(x-y)^2 .$$

Write down the general relation between the force $\underline{F}(\underline{r})$ and the potential energy $U(\underline{r})$ and use it to determine the force $\underline{F}(\underline{r})$ acting on the particle. **[3]**

Find the work done against the force $\underline{F}(\underline{r})$ when the particle moves from the point $(0,0)$ to the point $(1,1)$ along the line $x = y$. **[2]**

Write down the particle's equation of motion and show that it can be written as

$$\underline{\ddot{r}} = \underline{A}\,\underline{r} ,$$

where $\underline{A}$ is the 2×2 real matrix **[4]**

$$\underline{A} = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} .$$

Find the eigenvalues and eigenvectors of $\underline{\mathrm{A}}$. **[5]**

Use these eigenvalues and eigenvectors to obtain two uncoupled differential equations describing the motion. Solve these equations for the initial conditions $x = y = \frac{\mathrm{d}y}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{2}$ at $t = 0$. Give explicit solutions for the variables x and y. **[6]**

TURN OVER

5. In atomic units the Schrödinger equation for the hydrogen atom can be written as

$$\left(-\frac{1}{2}\nabla^2 - \frac{1}{r}\right)\psi = E\psi$$

where, in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\,.$$

By writing

$$\psi = R(r)\Theta(\theta)\Phi(\phi)$$

show that $\Phi(\phi)$ must satisfy the equation

$$\frac{d^2\Phi}{d\phi^2} = -m^2\Phi\,.$$

What are the solutions of this equation? Explain why m, the constant of separation, must take integer values. **[4]** **[3]**

Hence show that $R(r)$ must satisfy the radial equation

$$\frac{1}{2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + Rr + ERr^2 = \lambda R\,,$$

where λ is another constant of separation. Obtain the corresponding equation for $\Theta(\theta)$. **[6]**

For the special case of $\lambda = 0$, show that the radial equation can be written as

$$\frac{1}{2}\frac{d^2U}{dr^2} + \left[\frac{1}{r} + E\right]U = 0$$

where $U(r) = rR(r)$. Show that for large values of r **[3]**

$$U(r) = A\exp(\alpha r) + B\exp(-\alpha r)$$

where $\alpha = (-2E)^{\frac{1}{2}}$. How can this solution be simplified for bound states of the hydrogen atom? **[4]**

CONTINUED

6. (a) Any continuous function $f(x)$ in $-1 \leq x \leq 1$ can be expanded in terms of Legendre polynomials as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{for} \quad -1 \leq x \leq +1 \ .$$

Given the orthogonality relation

$$\int_{-1}^{+1} P_m(x) P_n(x) \, \mathrm{d}x = \delta_{mn} \frac{2}{2n+1} \ ,$$

derive a formula for the coefficients a_n. **[5]**

Given the first two Legendre polynomials

$$P_0(x) = 1 \ , \quad P_1(x) = x \ ,$$

find the first two coefficients a_0 and a_1 of the expansion of the function $\alpha \, \mathrm{e}^{\alpha|x|}$, where α is real. **[5]**

(b) The electrostatic potential of a charge Q located on the z axis at the point $(0, 0, d)$ (in Cartesian coordinates; note that d can be either positive or negative), reads, in spherical polar coordinates:

$$V(r, \theta, \phi) = \frac{Q}{4\pi\varepsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{d}{r} \right)^l P_l(\cos\theta) \ .$$

Add now a second charge $-Q$ on the z axis at point $(0, 0, -d)$. Write down an expression for the total potential V_T created by the charges Q and $-Q$ in spherical polar coordinates. **[4]**

Consider a charge q very far from the origin ($d \ll r$). By approximating the total potential V_T due to charges Q and $-Q$ to the first non-zero term of the expansion in Legendre polynomials, and using the expression of the gradient in spherical polars (given below), find the electrostatic force acting on the charge q in this case. **[6]**

The gradient operator in spherical polar coordinates is

$$\underline{\nabla} = \underline{\hat{e}}_r \frac{\partial}{\partial r} + \underline{\hat{e}}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \underline{\hat{e}}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \ .$$

TURN OVER

7. (a) Let $f(x)$ be a function of period π defined by

$$f(x) = \sin(x) \quad \text{for} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \; .$$

Is $f(x)$ an even or odd function?

Show that the Fourier expansion of $f(x)$ can be written as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(2nx)$$

where the coefficients b_n are given by **[4]**

$$b_n = \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \sin(2nx) \sin(x) \, \mathrm{d}x \; .$$

Determine the coefficients b_n and hence show that **[5]**

$$f(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{8n}{1-4n^2} \sin(2nx) \; .$$

Parseval's identity for a function $f(x)$ with general period $2L$ reads

$$\frac{1}{2L} \int_{-L}^{+L} [f(x)]^2 \, \mathrm{d}x = (a_0/2)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \; .$$

Apply Parseval's identity to prove **[3]**

$$\sum_{n=1}^{\infty} \frac{n^2}{(1-4n^2)^2} = \frac{\pi^2}{64} \; .$$

(b) The function $g(x)$ is defined by

$$\begin{aligned} g(x) &= \sin(x) \quad \text{for} \quad -l < x < +l \; , \\ g(x) &= 0 \quad \text{for} \quad |x| \geq l \; , \end{aligned}$$

where l is real and positive. The Fourier transform $\tilde{g}(k)$ of $g(x)$ is defined as

$$\tilde{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{-ikx} g(x) \, \mathrm{d}x \; .$$

Give the general way to obtain the original function $g(x)$ from its Fourier transform $\tilde{g}(k)$ (*i.e.*, to obtain the inverse Fourier transform of $\tilde{g}(k)$). **[3]**

Prove that, in the limit $l \to +\infty$, one has **[5]**

$$\lim_{l \to \infty} \tilde{g}(k) = i \sqrt{\frac{\pi}{2}} \left[\delta(k+1) - \delta(k-1) \right] \; ,$$

where δ stands for the Dirac delta function.

END OF PAPER

PHAS2246: Mathematical Methods III

2008/2009

Model solutions. A more detailed PROVISIONAL marking scheme is given here in square brackets in the right-hand margin.

1. (a) A vector field $\underline{F}(\underline{r})$ is conservative if and only if there exists a scalar potential $U(\underline{r})$ such that $\underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$ **(book work)**. **[1]**

   If $\underline{\nabla} \times \underline{F} = 0$ then the field $\underline{F}$ is conservative **(book work)**. The opposite implication ($\underline{F}$ conservative then $\underline{\nabla} \times \underline{F} = 0$) is also true. **[2]**

$$\begin{aligned} \underline{\nabla} \times \underline{F}_1 &= (\partial_y F_{1z} - \partial_z F_{1y})\underline{\hat{e}}_x + (\partial_z F_{1x} - \partial_x F_{1z})\underline{\hat{e}}_y + (\partial_x F_{1y} - \partial_y F_{1x})\underline{\hat{e}}_z = 0 \,, \\ \underline{\nabla} \times \underline{F}_2 &= -10z^4 \underline{\hat{e}}_y \,. \end{aligned}$$

   Hence, $\underline{F}_1$ is conservative but $\underline{F}_2$ is not. **[2]**

   The scalar potential $U(x, y, z)$ such that $\underline{F}_1 = -\underline{\nabla} U$ can be found by integrating the components of $\underline{F}_1$ with respect to the corresponding variables and by comparing the results obtained:

$$\begin{aligned} \frac{\partial U}{\partial x} &= -(2xy - z^5) \quad \Rightarrow \quad U = -\int (2xy - z^5)\,\mathrm{d}x + f_x(y,z) = -x^2 y + xz^5 + f_x(y,z) \,, \\ \frac{\partial U}{\partial y} &= -x^2 \quad \Rightarrow \quad U = -\int x^2\,\mathrm{d}y + f_y(x,z) = -x^2 y + f_y(x,z) \,, \\ \frac{\partial U}{\partial z} &= (5xz^4 + 1) \quad \Rightarrow \quad U = \int (5xz^4 + 1)\,\mathrm{d}z + f_z(x,y) = xz^5 + z + f_z(x,y) \,, \end{aligned}$$

   where the three functions f_x, f_y and f_z have to be determined by consistency. The only consistent option is $f_x(y,z) = z$, $f_y(x,z) = xz^5 + z$ and $f_z(x,y) = -x^2 y$, yielding **[4]**

$$U(x,y,z) = -x^2 y + xz^5 + z \,.$$

   As can be directly verified, the negative gradient of this potential U is equal to the field $\underline{F}_1$.

   (b) Evaluate the curl: **[2]**

$$\begin{aligned} \underline{\nabla} \times \underline{G} &= (\partial_y G_z - \partial_z G_y)\underline{\hat{e}}_x + (\partial_z G_x - \partial_x G_z)\underline{\hat{e}}_y + (\partial_x G_y - \partial_y G_x)\underline{\hat{e}}_z \\ &= -5\underline{\hat{e}}_z \,. \end{aligned}$$

   Because of Stokes' theorem, since the area of the square is 4 and the curl $\underline{\nabla} \times \underline{G}$ points towards the negative direction, one has **[3]**

$$\oint \underline{G}(x,y,z) \cdot \mathrm{d}\underline{r} = -5 \times 4 = -20 \,.$$

   (c) In polar coordinates:

$$x = \rho\cos\theta \,, \quad x^2 + y^2 = \rho^2 \quad \Rightarrow \quad H = x(x^2 + y^2) - z = \rho^3 \cos\theta - z$$

[1]

$$\Rightarrow \quad \nabla^2 H(\rho, \theta, z) = 9\rho\cos\theta - \rho\cos\theta = 8\rho\cos\theta\,.$$

[1]

Now, because the Laplacian $\nabla^2 H$ is just the divergence of the gradient $\underline{\nabla} H$, we can apply the divergence theorem and evaluate the surface integral by integrating $\nabla^2 H(\rho, \theta, z)$ over the volume of the cylinder: **[2]**

$$\int \underline{\nabla} H \cdot \mathrm{d}\underline{S} = \int_0^h \mathrm{d}z \int_0^R \mathrm{d}\rho \int_0^{2\pi} \rho\,\mathrm{d}\theta 8\rho\cos\theta = h\frac{8R^3}{3}\int_0^{2\pi} \cos\theta\,\mathrm{d}\theta = 0\,.$$

[2]

2. Look for a solution of the second-order differential equation

$$x\,\frac{d^2y}{dx^2} + (2-x)\,\frac{dy}{dx} + by = 0$$

Gives $p(x) = \frac{2-x}{x}$ and $q(x) = \frac{b}{x}$. **[1]**

This means the equation is singular at $x = 0$ *only*. **[1]**

At $x = 0$, $p_0 = 2$ and $q_0 = 0$, hence the indicial equation is written as

$$k(k-1) + 2k = 0$$

$$k^2 + k = 0$$

So $k = 0$ or -1. **[3]**

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ y' &= \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1}\,, \\ y'' &= \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)\, x^{n+k-2}\,. \end{aligned}$$

[2]

Inserting these into the equation, we obtain

$$\sum_{n=0}^{\infty} a_n \left[(n+k)(n+k-1)x^{n+k-1} + 2(n+k)\,x^{n+k-1} - (n+k)\,x^{n+k} + bx^{n+k}\right] = 0\,,$$

which can be grouped as

$$\sum_{n=0}^{\infty} a_n\,(n+k)(n+k+1)x^{n+k-1} = \sum_{n=0}^{\infty} a_n\,(n+k-b)\,x^{n+k}\,.$$

Rearranging the left hand side so that we get the same powers of x everywhere, we find

$$\sum_{n=-1}^{\infty} a_{n+1}\,(n+k+1)(n+k+2)x^{n+k} = \sum_{n=0}^{\infty} a_n\,(n+k-b)\,x^{n+k}\,.$$

[4]

The recurrence relation can be read off directly and gives

$$\frac{a_{n+1}}{a_n} = \frac{n+k-b}{(n+k+1)(n+k+2)}\,.$$

To check for convergence, we use the d'Alembert ratio test, which requires that

$$R = \left|\frac{a_{n+1}\,x^{n+1}}{a_n\,x^n}\right| < 1$$

in the $n \to \infty$ limit. From the recurrence relation, we see that

$$R \to \frac{|x|}{n} \quad \text{as } n \to \infty \, .$$

Clearly, in the limit of large n, we always have $R < 1$ so that the series converges for all values of x.
Or state that there are no poles in the complex plane apart from $x = 0$ so the solutions converges for all values of x. **[2]**

For $k = 0$ and $b = m$, a positive integer, the recurrence relation becomes

$$\frac{a_{n+1}}{a_n} = \frac{n - m}{(n+1)(n+2)} \, .$$

The right hand side vanishes when $n = m$ so that $a_{m+1} = 0$. By repeated use of the recurrence relation, all the subsequent terms are then also zero. (This last bit is crucial.) **[3]**

For $b = m = 2$

$$a_1 = \frac{-2}{1 \times 2} a_0 = -a_0$$

$$a_2 = \frac{1-2}{2 \times 3} a_1 = \frac{a_0}{6}$$

$$a_3 = 0$$

[2]

$$y = (\frac{x^2}{6} - x + 1) a_0$$

$$y' = (\frac{x}{3} - 1) a_0$$

$$y'' = \frac{1}{3} a_0$$

which satisfies the equation. **[2]**

3. A Hermitian matrix is one for which $\underline{A}^\dagger = (\underline{A}^T)^* = (\underline{A}^*)^T = \underline{A}$ **[1]**

Consider the eigenvalue equation

$$\underline{H}\,\underline{X} = \lambda\,\underline{X}\,,$$

Take its Hermitian conjugate:

$$\begin{aligned} (\underline{H}\,\underline{X})^\dagger &= (\lambda\,\underline{X})^\dagger\,, \\ \underline{X}^\dagger\,\underline{H}^\dagger = \underline{X}^\dagger\,\underline{H} &= \lambda^*\,\underline{X}^\dagger\,. \end{aligned} \quad (1)$$

Multiply eq. (1) from the right by $\underline{X}$

$$\underline{X}^\dagger\,\underline{H}\,\underline{X} = \lambda^*\,\underline{X}^\dagger\,\underline{X}\,. \quad (2)$$

Go back to first Eq. and multiply it on the left by $\underline{X}^\dagger$

$$\underline{X}^\dagger\,\underline{H}\,\underline{X} = \lambda\,\underline{X}^\dagger\,\underline{X}\,. \quad (3)$$

The left hand sides of Eqs. (2) and (3) are identical and so the right hand sides have to be as well;

$$(\lambda^* - \lambda)\,\underline{X}^\dagger\,\underline{X} = 0\,. \quad (4)$$

But since all $\underline{X}^\dagger\,\underline{X} = X^{\,2}$ are non-zero

$$\lambda_i^* - \lambda_i = 0\,, \quad (5)$$

which means that all the eigenvalues are **real** **(adaptation of bookwork)**. **[4]**

Eigenvectors for non-degenerate eigenvalues are orthogonal. **[2]**

$\underline{A}$ is real and symmetric and hence Hermitian. **[1]**

The trace of $\underline{A}$ is 5+11+5=21. **[1]**

The characteristic equation is given by

$$|\underline{A} - \lambda \underline{I}| = \begin{vmatrix} 5-\lambda & -5 & 1 \\ -5 & 11-\lambda & -5 \\ 1 & -5 & 5-\lambda \end{vmatrix}\,.$$

[1]

To verify that $\lambda_1 = 16$, evaluate

$$\begin{vmatrix} -11 & -5 & 1 \\ -5 & -5 & -5 \\ 1 & -5 & -11 \end{vmatrix}\,.$$

This equals zero since adding row 1 and row 3 gives twice row 2. **[1]**

To verify that $\underline{v}_1$ is the associated eigenvector,

$$(\underline{A}-\lambda_1\underline{I})\,\underline{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} -11 & -5 & 1 \\ -5 & -5 & -5 \\ 1 & -5 & -11 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -11+10+1 \\ -5+10-5 \\ 1+10-11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

as required. It is a normalised eigenvector by inspection. **[1]**

To verify that $\lambda_2 = 4$, evaluate

$$\begin{vmatrix} 1 & -5 & 1 \\ -5 & 7 & -5 \\ 1 & -5 & 1 \end{vmatrix}.$$

This equals zero since the first and third rows are the same. **[1]**

The sum of the eigenvalues equals the trace of the matrix, so

$$\lambda_3 = 21 - 16 - 4 = 1$$

Other methods of demonstrating eigenvalues and eigenvector acceptable. **[1]**

For the second eigenvalue,

$$(\underline{A} - \lambda_2\underline{I})\,\underline{v}_2 = \begin{pmatrix} 1 & -5 & 1 \\ -5 & 7 & -5 \\ 1 & -5 & 1 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $v_{12} = -v_{32}$, hence $v_{22} = 0$. The normalised eigenvector is therefore (to within a phase). **[2]**

$$\underline{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

[1]

Similarly for the third eigenvalue,

$$(\underline{A} - \lambda_3\underline{I})\,\underline{v}_3 = \begin{pmatrix} 4 & -5 & 1 \\ -5 & 10 & -5 \\ 1 & -5 & 4 \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \\ v_{32} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that $v_{12} = v_{32}$ and hence $v_{22} = v_{12} = v_{32}$. The normalised eigenvector is therefore (to within a phase). **[2]**

$$\underline{v}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

[1]

4. Relation between force $\underline{F}$ and potential U **(book work)**: **[1]**

$$\underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$$

$$\Rightarrow \quad \underline{F}(\underline{r}) = -(2x - \sqrt{2}y)\underline{\hat{e}}_x - (y - \sqrt{2}x)\underline{\hat{e}}_y \, .$$

The force field is conservative (admits a potential U). The work W done against the force is just given by the difference in the potential at the initial and final point: **[2]**

[2]

$$W = U(1,1) - U(0,0) = \frac{3}{2} - \sqrt{2} - 0 = \frac{3}{2} - \sqrt{2} \, .$$

Newton's equation of motion: **[2]**

$$\underline{F}(\underline{r}) = m\underline{\ddot{r}} = \underline{\ddot{r}} \quad \Rightarrow \quad \underline{\ddot{r}} = -(2x - \sqrt{2}y)\underline{\hat{e}}_x - (y - \sqrt{2}x)\underline{\hat{e}}_y \, ,$$

which can be written in terms of components and matrices as: **[2]**

$$\underline{\ddot{r}} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -2x + \sqrt{2}y \\ -y + \sqrt{2}x \end{pmatrix} = \begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underline{A}\,\underline{r} \, .$$

Characteristic equation of $\underline{A}$: **[2]**

$$\lambda^2 + 3\lambda = 0$$

$\Rightarrow$ eigenvalues are 0 and -3. **[1]**

Eigenvector related to 0:

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}} \, .$$

Up to normalisation, the eigenvector $\underline{r_0}$ corresponding to the eigenvalue 0 is **[1]**

$$\underline{r_0} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \end{pmatrix} \, .$$

Eigenvector related to -3:

$$\begin{pmatrix} -2 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} -3a \\ -3 \end{pmatrix} \quad \Rightarrow \quad a = -\sqrt{2} \, .$$

Up to normalisation, the eigenvector $\underline{r_{-3}}$ corresponding to the eigenvalue -3 is **[1]**

$$\underline{r_{-3}} = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} \, .$$

In the new variables $\tilde{x} = x/\sqrt{2} + y$ and $\tilde{y} = -\sqrt{2}x + y$, dictated by the eigenvectors above, the second-order differential equations of motion decouple as **[2]**

$$
\begin{aligned}
\ddot{\tilde{x}} &= 0 \, , \\
\ddot{\tilde{y}} &= -3\tilde{y} \, ,
\end{aligned} \tag{6}
$$

with general solutions **[1]**

$$
\begin{aligned}
\tilde{x} &= At + B \, , \\
\tilde{y} &= C\sin(\sqrt{3}t) + D\cos(\sqrt{3}t) \, .
\end{aligned} \tag{7}
$$

The requested initial conditions read, in terms of the new variables:

$$
\tilde{x}(0) = \tilde{y}(0) = 0 \, , \quad \dot{\tilde{x}} = 1 \, , \quad \dot{\tilde{y}} = -2 \, ,
$$

which imply

$$
B = 0 \, , \quad D = 0 \, , \quad A = 1 \, , \quad C = -\frac{2}{\sqrt{3}} \, , .
$$

The requested solutions are thus **[1]**

$$
\begin{aligned}
\tilde{x} &= t \, , \\
\tilde{y} &= -\frac{2}{\sqrt{3}}\sin(\sqrt{3}t) \, ,
\end{aligned}
$$

which can be expressed in terms of the original variables x and y by noting that: **[1]**

$$
\begin{aligned}
x &= \frac{\sqrt{2}}{3}(\tilde{x} - \tilde{y}) \, , \\
y &= \frac{\sqrt{2}}{3}(\sqrt{2}\tilde{x} + \frac{1}{\sqrt{2}}\tilde{y}) \, .
\end{aligned}
$$

Finally: **[1]**

$$
\begin{aligned}
x &= \frac{\sqrt{2}}{3}(t + \frac{2}{\sqrt{3}}\sin(\sqrt{3}t)) \, , \\
y &= \frac{2}{3}(t - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)) \, .
\end{aligned}
$$

5. **(This question is similar to material covered in the lectures where they did separation of variables for Laplace's equation)**

$$\psi(r,\,\theta,\,\phi) = R(r)\times\Theta(\theta)\times\Phi(\phi)\,.$$

$$\Theta\,\Phi\,\frac{1}{2r^2}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)+R\,\Phi\,\frac{1}{2r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+R\,\Theta\,\frac{1}{2r^2\sin^2\theta}\left(\frac{d^2\Phi}{d\phi^2}\right)+(\frac{1}{r}+E)R\,\Theta\,\Phi = 0.$$

Divide by $R\,\Theta\,\Phi$ and multiply by $2r^2\sin^2\theta$ **[2]**

$$\frac{\sin^2\theta}{R}\,\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)+\frac{1}{\Theta}\,\sin\theta\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+2r\sin^2\theta(1+Er)+\frac{1}{\Phi}\,\left(\frac{d^2\Phi}{d\phi^2}\right)=0\,.$$

First 3 terms here depend upon r and θ but last is a function purely of ϕ. Since r, **[1]** θ and ϕ are independent variables, the third term must be some constant, denoted by $-m^2$. **[1]**

$$\frac{d^2\Phi}{d\phi^2} = -m^2\,\Phi\,,$$

which has solutions $e^{\pm im\phi}$ or, alternatively, $\cos m\phi$ and $\sin m\phi$. When ϕ increases by 2π, the solution returns to the same point; expect same physical solution. Thus $\Phi(\phi+2\pi)=\Phi(\phi)$. Can only be accomplished if m is a real integer. Then $\Phi(\phi)$ is clearly a periodic function. **[3]**

The remainder of the equation can be manipulated (divide by $2\sin^2\theta$ and rearrange) into

$$\frac{1}{2R}\,\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)+r(1+Er)=\frac{1}{2}\left[\frac{m^2}{\sin^2\theta}-\frac{1}{\Theta}\,\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)\right]\,.$$

Left hand side is function only of r, while right hand side depends only on θ. **[2]** Means that both sides must be equal to some constant, denote by λ. Results in **[2]** two ordinary DEs:

$$\begin{aligned}\frac{1}{2}\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right)+r(1+Er)R &= \lambda\,R\,,\\ \frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right)+\left(2\lambda\,\sin\theta-\frac{m^2}{\sin\theta}\right)\,\Theta &= 0\,.\end{aligned}$$

[2]

Let $U=rR$ and $\lambda=0$.

$$R' = -r^{-2}U + r^{-1}U'$$

[1]

$$r^2R' = -U + rU'$$

$$\frac{d}{dr}\left(r^2\,\frac{dR}{dr}\right) = -U' + U' + rU''$$

[1]

$$rU'' + 2U + 2ErU = 0$$

gives [1]

$$\frac{1}{2}\frac{d^2U}{dr^2} + \left[\frac{1}{r} + E\right]U = 0$$

For large r, $\frac{1}{r}$ tends to zero so

$$U'' = -2EU$$

which has solution for $\alpha = (-2E)^{\frac{1}{2}}$ [1]

$$U(r) = A\exp(\alpha r) + B\exp(-\alpha r).$$

For bound states this solution must be normalisable, however $\exp(\alpha r)$ is infinite at $r = \infty$ so cannot be normalised, therefore $A = 0$. [1] [2]

6. (a) Multiplying the expansion of $f(x)$ in terms of Fourier polynomials by $P_m(x)$ and integrating over x yields **[2]**

$$\int_{-1}^{+1} P_m(x) f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-1}^{+1} P_m(x) P_n(x)\,\mathrm{d}x\ .$$

Substituting the orthogonality relation into the RHS of the previous equation gives: **[2]**

$$\int_{-1}^{+1} P_m(x) f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \delta_{mn} \frac{2}{2n+1} = a_m \frac{2}{2m+1}\,.$$

Now, rearranging terms and renaming the label gives the general formula for the coefficients a_n **(book work)** **[1]**

$$a_n = \frac{2n+1}{2} \int_{-1}^{+1} P_n(x) f(x)\,\mathrm{d}x\ .$$

Notice that $\alpha\,\mathrm{e}^{\alpha|x|}$ is even. Applying the formula above, splitting the integration domain and noting that $P_n(x)$ is even or odd for even or odd n, one has: **[2]**

$$a_0 = \frac{1}{2}\int_{-1}^{+1} \alpha\,\mathrm{e}^{\alpha|x|}\,\mathrm{d}x = \int_{0}^{+1} \alpha\,\mathrm{e}^{\alpha x}\,\mathrm{d}x = [\mathrm{e}^{\alpha x}]_0^1 = \mathrm{e}^{\alpha} - 1\,.$$

$a_1 = 0$ because $P_1(x)\alpha\,\mathrm{e}^{\alpha|x|}$ is odd. **[3]**

(b) If the second charge is added, the total potential is: **[4]**

$$\begin{aligned} V_T(r,\theta,\phi) &= \frac{Q}{4\pi\varepsilon_0 r}\left[\sum_{l=0}^{\infty}\left(\frac{d}{r}\right)^l P_l(\cos\theta) - \sum_{l=0}^{\infty}\left(\frac{-d}{r}\right)^l P_l(\cos\theta)\right] \\ &= \frac{Q}{4\pi\varepsilon_0 r}\sum_{l=0}^{\infty}\left(\frac{d}{r}\right)^l P_l(\cos\theta) + \sum_{l=0}^{\infty}(-1)^{l+1}\left(\frac{d}{r}\right)^l P_l(\cos\theta) \\ &= \frac{2Q}{4\pi\varepsilon_0 r}\sum_{l\,\mathrm{odd}}^{\infty}\left(\frac{d}{r}\right)^l P_l(\cos\theta) \end{aligned}$$

(the even terms in the two expansions cancel each other out). Hence, $l = 1$ is the first non-vanishing term ("dipole" term) and, for $d \ll r$, the potential can be approximated as **[2]**

$$V_T(r,\theta,\phi) \simeq \frac{Q}{2\pi\varepsilon_0 r}\frac{d}{r}P_1(\cos\theta) = \frac{Qd}{2\pi\varepsilon_0 r^2}\cos\theta$$

The resulting electrostatic force is thus given by **[4]**

$$\underline{F} = -q\underline{\nabla}V_T = \frac{Qqd}{2\pi\varepsilon_0}\left(-\partial_r\frac{\cos\theta}{r^2}\,\underline{\hat{e}}_r - \frac{1}{r}\partial_\theta\frac{\cos\theta}{r^2}\,\underline{\hat{e}}_\theta\right) = \frac{Qqd}{2\pi\varepsilon_0 r^3}(2\cos\theta\,\underline{\hat{e}}_r + \sin\theta\,\underline{\hat{e}}_\theta)\,.$$

This represents the interaction of an electric dipole with a charge. Notice that this force is one order weaker (proportional to $1/r^3$) than that of a single charge (as the net charge of the present configuration vanishes and thus there is no "monopole" contribution).

7. (a) The function $f(x)$ is odd because $\sin x$ is odd. **[1]**

Fourier series of a function with generic period $2L$ **(book work)**: **[1]**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\frac{\pi}{L}x) + \sum_{n=1}^{\infty} b_n \sin(n\frac{\pi}{L}x) \ ,$$

where the coefficients of the expansion are given by **(book work)**: **[1]**

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{+L} \cos(n\frac{\pi}{L}x) f(x)\,\mathrm{d}x \,, \\ a_n &= \frac{1}{L}\int_{-L}^{+L} \cos(n\frac{\pi}{L}x) f(x)\,\mathrm{d}x \,. \end{aligned}$$

All the a_n's vanish because $f(x)$ is odd. Hence, replacing L with $\pi/2$ yields **[1]**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(2nx) \ ,$$

with

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} \sin(2nx) f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_{-\pi/2}^{+\pi/2} \sin(2nx)\sin(x)\,\mathrm{d}x \ .$$

To solve the previous integral, apply the goniometric identity:

$$\sin(\alpha)\sin(\beta) = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2} \ ,$$

so that

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \cos((2n-1)x) - \cos((2n+1)x)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left(\left[\frac{\sin((2n-1)x)}{2n-1}\right]_{-\pi/2}^{+\pi/2} - \left[\frac{\sin((2n+1)x)}{2n+1}\right]_{-\pi/2}^{+\pi/2}\right) \ . \end{aligned}$$

Note now that $\sin((2n+1)\pi/2) = (-1)^n$ (and, thus, $\sin((2n-1)\pi/2) = (-1)^{n+1}$): **[2]**

$$b_n = \frac{(-1)^n}{\pi}\left(-\frac{2}{2n-1} - \frac{2}{2n+1}\right) = \frac{(-1)^n}{\pi}\frac{8n}{1-4n^2} \ .$$

Substituting into the Fourier series: **[2]**

[1]

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{8n}{1-4n^2}\sin(2nx) \ .$$

Applying Parseval's identity to our case:

$$\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \sin^2(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi/2}^{+\pi/2} 1 - \cos(2x)\,\mathrm{d}x = \frac{1}{2} - 0 = \frac{1}{2}$$

[2]

$$\Rightarrow \quad \frac{1}{2} = \frac{1}{2}\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{64n^2}{(1-4n^2)^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{n^2}{(1-4n^2)^2} = \frac{\pi^2}{64}\,.$$

[1]

(b) In general one has **(book work)**: [3]

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\tilde{g}(k)\,\mathrm{e}^{ikx}\,\mathrm{d}k\ .$$

Substituting $\tilde{g}(k) = i\sqrt{\frac{\pi}{2}}\,[\delta(k+1)-\delta(k-1)]$ into the RHS of the previous equation, and using the properties of the delta function, yields [5]

$$\frac{i}{2}\int_{-\infty}^{+\infty}[\delta(k+1)-\delta(k-1)]\,\mathrm{e}^{ikx}\,\mathrm{d}k = \frac{1}{2i}\left[\mathrm{e}^{ix}-\mathrm{e}^{-ix}\right] = \sin x = \lim_{l\to\infty} g(x)\,,$$

which proves the requested identity. Note that the limit $l \to \infty$ is essential, as $g(x) = \sin x$ holds *everywhere* only in this limit.