UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS2246**
ASSESSMENT PATTERN	:	**PHAS2246A**
MODULE NAME	:	**Mathematical Methods III**
DATE	:	**28-Apr-08**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 30 Minutes**

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TURN OVER

ALL questions, including just parts of questions, may be attempted.
Credit will given for all correct work done.

For guidance: A student should aim to correctly answer the equivalent of five questions in the time available.

Numbers in square brackets show the provisional allocation of marks per sub-section of the question.

1. (a) The temperature $T(x,t)$ in a metal bar at time t satisfies the equation

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{2}\frac{\partial T}{\partial t} = 0\,,$$

where the coordinate x lies in the range $0 \le x \le 2\pi$.

By separating the variables x and t, show that one solution for T is given by

$$T(x,t) = \{A\cos(\lambda x) + B\sin(\lambda x)\}\, e^{-2\lambda^2 t}\,,$$

where A, B and λ are constants. **[7 marks]**

If the temperatures at $x = 0$ and $x = \pi$ are kept at 0 K, show that the temperature in the bar must have the form

$$T(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)\exp(-2n^2 t)\,.$$

where n is a positive interger. **[3 marks]**

(b) Consider the function

$$f(x) = \frac{\lambda}{2}\,\mathrm{e}^{-\lambda|x|} \quad \text{for } -\infty < x < \infty\,,$$

where λ is real and positive. Is the function $f(x)$ even or odd? **[1 mark]**

The function $g(k)$ is said to be the Fourier transform of $f(x)$: give an expression linking the two functions. **[2 marks]**

By splitting the integration into two parts, from $-\infty$ to 0 and from 0 to $+\infty$, or otherwise, determine the Fourier transform $g(k)$ of $f(x)$. **[7 marks]**

2. If $\underline{A}$ and $\underline{B}$ are both matrices of dimension $n \times n$, prove that

$$(\underline{B}\,\underline{A})^T = \underline{A}^T \underline{B}^T \,.$$

Hence show that $\underline{A}^T \underline{A}$ is symmetric. **[4 marks]**

A matrix is given by

$$\underline{B} = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{pmatrix} .$$

Find the inverse matrix $\underline{B}^{-1}$. **[7 marks]**

For a general, non-singular matrix $\underline{A}$ prove that

$$\underline{A}^{-1} = (\underline{A}^T \underline{A})^{-1} \underline{A}^T$$

[2 marks]

Verify this result explicitly by using the matrix $\underline{B}$ given above. **[7 marks]**

3. The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & -3 \\ 3 & -3 & -2 \end{pmatrix} .$$

Verify that the eigenvalues of this matrix are $\lambda_1 = 4$, $\lambda_2 = 3$ and $\lambda_3 = -5$. Show that the *normalised* eigenvector associated with λ_1 can be written

$$\underline{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} .$$

[7 marks]

Find the other two normalised eigenvectors $\underline{v}_2$ and $\underline{v}_3$, associated with eigenvalues λ_2 and λ_3 respectively. **[6 marks]**

Show that these eigenvectors are mutually orthogonal and that, up to a possible overall sign, **[3 marks]**

$$\underline{v}_3 = \pm(\underline{v}_1 \times \underline{v}_2) \,.$$

[2 marks]

Explain the origin of these last two results. **[2 marks]**

4. A matrix $\underline{H}$ is described as Hermitian, what property does it have? Another matrix $\underline{R}$ is described as Unitary, what property does it have? **[4 marks]**

The (2×2) matrix $\underline{H}$ has elements $H_{11} = H_{22} = -\alpha$, $H_{12} = H_{21} = \beta$, where α and β are real positive constants with $\alpha > \beta$. If $\underline{H}$ is Hermitian, what properties must the eigenvalues and eigenvectors have? **[2 marks]**

Determine the eigenvalues and eigenvectors of $\underline{H}$ and hence write down the unitary matrix $\underline{R}$ such that $\underline{R}\ \underline{H}\underline{R}^{-1}$ is diagonal. **[5 marks]**

The following coupled differential equations:

$$\ddot{x}_1 = -5x_1 + 4x_2 \,,$$

$$\ddot{x}_2 = 4x_1 - 5x_2 \,,$$

where

$$\ddot{x}_i = \frac{d^2 x_i}{dt^2} \,,$$

can be written in matrix form as:

$$\ddot{\mathbf{x}} = \underline{A}\,\mathbf{x} \,.$$

Give the form for $\underline{A}$ and hence find its eigenvalues and eigenvectors. **[4 marks]**

Use these eigenvalues and eigenvectors to write two uncoupled differential equations. Hence give a general solution for these equations subject to the initial condition that $x_1 = x_2 = 0$ at $t = 0$. **[5 marks]**

5. Consider the following second order linear differential equation

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + bx\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - b)y = 0 \,, \qquad (1)$$

where $b \geq 0$.

What is meant by saying a differential equation is even? Is Eq. (1) even? Write the equation in the form $y'' + p(x)y' + q(x)y = 0$. For what values of x are $p(x)$ and $q(x)$ singular? What kind of singularities are these? **[5 marks]**

The equation admits series solutions of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \,.$$

Obtain the indicial equation and show that the allowed values of k are $+1$ and $-b$. **[4 marks]**

Derive the recurrence relation for the series solution. **[5 marks]**

Determine the radius of convergence of the series solution for $k = 1$. Show that independent even and odd series solutions exist for $k = 1$. **[4 marks]**

What are the odd and even solutions for the case where $k = -b = 0$? **[2 marks]**

6. Legendre polynomials satisfy the following orthogonality relation:

$$\int_{-1}^{+1} P_m(x)P_n(x)\,\mathrm{d}x = \frac{2}{2m+1}\delta_{mn}\,. \tag{1}$$

Explain the significance of such a relation with respect to the possibility of expanding a continous function $f(x)$ in $-1 \leq x \leq +1$ as a sum of Legendre polynomials:

[1 mark]

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad \text{for} \; -1 \leq x \leq +1\,. \tag{2}$$

Give a general formula that can be used to evaluate the coefficient a_n? **[2 marks]**

$P_m(x)$ and $P_n(x)$ also satisfy the Legendre equations

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)P_n'(x)\Big] = -n(n+1)P_n(x)\,, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[(1-x^2)P_m'(x)\Big] = -m(m+1)P_m(x)\,. \tag{4}$$

Multiplying Eq. (3) by $P_m(x)$ and Eq. (4) by $P_n(x)$, subtracting the two equations, and integrating over x from -1 to $+1$, derive Eq. (1) for the case $m \neq n$. **[6 marks]**

Given the expression of the first 3 Legendre polynomials

$$P_0(x) = 1\,, \quad P_1(x) = x\,, \quad P_2(x) = \frac{1}{2}(3x^2-1)\,,$$

determine the expansion of the function (x^2+3x-1) in terms of Legendre polynomials. **[3 marks]**

Consider now the function:

$$f(x) = \frac{2}{\sqrt{5-4x}}\,.$$

Between -1 and $+1$, such a function can be expanded as a sum of Legendre polynomials according to Eq. (2). Find out the first two coefficients of the expansion a_0 and a_1. **[4 marks]**

Using the generating function

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad \text{for } |t| < 1$$

and choosing a suitable t, determine all the coefficients of the expansion of $f(x)$ in terms of Legendre polynomials. **[4 marks]**

7. Consider a periodic function of period 2π given by the sum of two periodic functions with the same period 2π:

$$g(x) = h(x) + j(x) \ .$$

Show that the Fourier coefficients of $g(x)$ are given by the sums of the corresponding Fourier coefficients of $h(x)$ and $j(x)$ (that is, the Fourier expansion of $g(x)$ is just the sum of the expansions of $h(x)$ and $j(x)$). **[3 marks]**

Let $f(x)$ be a periodic function of period 2π defined, between $-\pi$ and $+\pi$, as

$$f(x) = \cos\left(\frac{x}{2}\right) + \frac{1}{10}\sin(5x) \ .$$

Show that the Fourier series of $f(x)$ is given by **[7 marks]**

$$f(x) = \frac{2}{\pi} + \frac{1}{10}\sin(5x) + \sum_{n=1}^{\infty} \frac{4(-1)^n}{(1-4n^2)\pi}\cos(nx) \ .$$

Evaluating the function in $x = 0$, show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{1-4n^2} = \frac{\pi - 2}{4} \ .$$ **[4 marks]**

Apply Parseval's identity to prove that

$$\sum_{n=1}^{\infty} \frac{1}{(1-4n^2)^2} = \frac{\pi^2 - 8}{16} \ .$$ **[6 marks]**

The following trigonometric identity can be assumed:

$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos((a+b)x) + \cos((a-b)x)\right] \ .$$

END OF PAPER