UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **PHAS2246**

MODULE NAME : **Mathematical Methods III**

DATE : **14-May-07**

TIME : **10:00**

TIME ALLOWED : **2 Hours 30 Minutes**

2006/07-PHAS2246A-001-EXAM-118

TURN OVER

Answer FIVE questions only.

Numbers in square brackets show the provisional allocation of marks per subsection of the question.

1. (a) Show that if for two matrices $\underline{A}$ and $\underline{B}$ the product $\underline{AB}$ is defined that

$$(\underline{AB})^T = \underline{B}^T \underline{A}^T$$

[2 mark]

(b) Given are the matrices

$$\underline{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} \qquad \underline{B} = \begin{pmatrix} -1 & 0 \\ 4 & 2 \\ 3 & -1 \end{pmatrix} \qquad \underline{C} = \begin{pmatrix} 7 & -1 & 2 \end{pmatrix}$$

Give $\underline{A}^T$, $\underline{B}^T$ and $\underline{C}^T$. **[3 marks]**

(c) Which of the following matrix products are possible? Evaluate the ones which are possible:
$\underline{AB}$, $\underline{AB}^T$, $\underline{BA}$, $\underline{B}^T\underline{A}^T$, $\underline{AC}$, $\underline{CA}$, $\underline{CB}$, $\underline{B}^T\underline{C}^T$, $\underline{CC}$. **[9 marks]**

(d) Calculate $\underline{A}^{-1}$. **[6 marks]**

2. The real quadratic form F in three dimensions is given by:

$$F = 2x^2 - 8xy + 2y^2 + 4z^2 = 0 \,,$$

(a) Write down the matrix $\underline{A}$ so that F is given by

$$F = \underline{v}^T \underline{A} \underline{v} = 0 \,,$$

with $\underline{v}^T = \begin{pmatrix} x & y & z \end{pmatrix}$. **[2 marks]**

(b) Find the three different eigenvalues of $\underline{A}$ by writing the characteristic equation in the form

$$(p - \lambda) \left\{ (q - \lambda)^2 - r \right\} = 0$$

and calculating the values of p, q and r. Calculate the three corresponding normalized eigenvectors. **[12 marks]**

(c) Evaluate the transformation matrix $\underline{S}$, for which $\underline{S}^T \underline{A} \underline{S}$ is diagonal. **[2 marks]**
Set $\underline{u} = \underline{S}^T \underline{v}$ and write the quadratic form F in the new variables $\underline{u}^T = \begin{pmatrix} \tilde{x} & \tilde{y} & \tilde{z} \end{pmatrix}$. **[4 marks]**

3. $\underline{f}$ and $\underline{g}$ are real vector fields in three dimensions and U and V are real scalar fields in three dimensions.

   (a) Define the terms *div*, *grad* and *curl* in terms of the operator $\underline{\nabla}$, real vector field $\underline{f}$ and scalar field U. **[3 marks]**

   (b) Proof $\underline{\nabla} \times (\underline{\nabla} U) = 0$ and $\underline{\nabla} \cdot (\underline{\nabla} \times \underline{f}) = 0$ **[4 marks]**
   and then show that
   $$\underline{f} = \frac{r^2 - 2}{r^4}\underline{r} \,,$$
   with $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $r = |\underline{r}|$,
   can be written as a gradient field ($\underline{f} = grad\, U$) **[5 marks]**

   (c) Show $\underline{\nabla} \cdot (U \underline{\nabla} V) = (\underline{\nabla} U) \cdot (\underline{\nabla} V) + U \underline{\nabla} \cdot (\underline{\nabla} V)$. **[4 marks]**

   (d) Show $\underline{\nabla} \cdot (\underline{f} \times \underline{g}) = \underline{g} \cdot (\underline{\nabla} \times \underline{f}) - \underline{f} \cdot (\underline{\nabla} \times \underline{g})$. **[4 marks]**

4. Solve
$$2(x^2 + x^3)\frac{d^2y}{dx^2} - (x - 3x^2)\frac{dy}{dx} + y = 0 \,,$$
with a general series solution. Write the differential equation in the general form
$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \,.$$
Evaluate the singular points of the differential equation. **[2 marks]** The equation has a series solution of the form
$$y = \sum_{n=0}^{\infty} a_n x^{n+k} \,.$$
Write down the indicial equation and show that $k = \frac{1}{2}$ and $k = 1$. **[3 marks]** Show that the recursion relations are given by
$$a_{n+1} = -a_n \,.$$ **[8 marks]**

Give the radius of convergence of these series. **[2 marks]** Calculate the first 4 terms of the series solution and show that the general solution can be written in the form
$$y(x) = \frac{Ax + B\sqrt{x}}{1 + x} \,.$$ **[5 marks]**

5. The function $f(x)$ is defined on the intervall $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ with

$$f(x) = 2\cos x$$

(a) Is $f(x)$ and even or odd function? **[1 mark]**

(b) The Fourier expansion is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \ ,$$

with $-L \leq x \leq L$. Show that the Fourier coefficients of $f(x) = 2\cos x$ with $L = \frac{\pi}{4}$ are given by

$$\begin{aligned} a_n &= \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(x)\cos(4nx)\, dx \\ b_n &= 0 \ . \end{aligned}$$

[7 marks]

(c) Evaluate the coefficients a_n and show

$$f(x) = \frac{4\sqrt{2}}{\pi} + \frac{8\sqrt{2}}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{1}{1-16n^2}\cos 4nx \ .$$

[8 marks]

Hints:
$\int \cos ax \cos bx \, dx = \frac{1}{2}\left(\frac{\sin(a-b)x}{a-b} + \frac{\sin(a+b)x}{a+b}\right)$
and
$\sin(a+b) = \sin a \cos b + \cos a \sin b$.

(d) By considering $f(x)$ at $x = \pi/4$ calculate the value of the series

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}$$

[4 marks]

6. (a) Use

$$nP_n(x) = (2n-1)xP_{n-1}(x) - (n-1)P_{n-2}(x)$$

with $P_0(x) = 1$ and $P_1(x) = x$ to calculate $P_2(x)$ and $P_3(x)$. Then express

i. $3x^2 + x - 1$

ii. $x - x^3$

in terms of a finite series of Legendre polynomials. [6 marks]

(b) The generating function $g(x,t)$ is related to the Legendre polynomials via

$$g(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \ . \qquad (1)$$

Show

$$(x-t)\frac{\partial g}{\partial x} = t\frac{\partial g}{\partial t} \qquad (2)$$

[4 marks]

(c) By substituting the series from equation (1) into equation (2) show that

$$xP_n'(x) - P_{n-1}'(x) = nP_n(x) \qquad (3)$$

where the prime denotes the derivative with respect to x. [5 marks]

(d) Differentiate

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x)$$

with respect to x and eliminate P_{n-1}' with the help of equation (3). What is the resulting equation? [5 marks]

7. The Schrödinger equation for a particle of mass m in a one dimensional potential $V(x)$ is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

(a) If you write $\Psi(x,t) = F(x) \times T(t)$ show that the solution of the differential equation is of the form

$$T(t) = Ce^{-iEt/\hbar}$$

[3 marks]

(b) Show, that for zero potential ($V(x) \equiv 0$), the solution is given by

$$\Psi(x,t) = \{A\cos kx + B\sin kx\}\, e^{-iEt/\hbar} \qquad (4)$$

[2 marks]

Further show that k and E are related by

$$k^2 = \frac{2m}{\hbar^2}E$$

[1 mark]

(c) Assume now that $V = 0$ for $0 \le x \le l$ and $\Psi(x,t) = 0$ at $x = 0$ and $x = l$ for all times t. Show that the general solution fullfilling these boundary conditions is

$$\Psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{-iE_nt/\hbar}$$

Give E_n as a function of n. **[8 marks]**

(d) $|\Psi|^2$ is the probability of finding a particle at position x. Show that in order to to ensure

$$\int_0^l |\Psi|^2\, dx = 1\ .$$

the coefficients B_n have to obey

$$\sum_{n=0}^{\infty} |B_n|^2 = \frac{2}{l}$$

[6 marks]

Hint: $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \pi\delta_{nm}$.

END OF PAPER