UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:–

B.Sc. *M.Sci.*

Mathematics M14B: Mathematical Methods 2

COURSE CODE : **MATHM14B**

UNIT VALUE : **0.50**

DATE : **28–APR–06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06–C1042–3–220

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.
   (b) Find the Fourier series of $f(x) = \text{sign}(x)$ on $(-\pi, \pi)$.
   (c) State and prove Parseval's identity for a function on $(-\pi, \pi)$.
   (d) Hence, or otherwise, prove that

$$\frac{\pi^2}{8} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2}.$$

2. (a) Using subscript notation, what is the expression for

$$\varepsilon_{ijk}\varepsilon_{klm}$$

   in terms of δ_{il}, δ_{jm}, etc.?
   (b) Using subscript notation, prove

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\,\text{curl}\,\mathbf{A}) + \mathbf{A}\times(\,\text{curl}\,\mathbf{B}).$$

3. (a) State Stokes' theorem carefully.
   (b) Show that Stokes' theorem implies that an integral of the following form is independent of path C, provided that the end points of C are fixed:

$$\int_C \text{grad}\,\phi\cdot d\mathbf{r},$$

   where ϕ is assumed to be smooth and defined everywhere.
   (c) Hence or otherwise find

$$\int_C \left(\frac{\mathbf{r}}{|\mathbf{r}|^3} + x\mathbf{i}\right)\cdot d\mathbf{r},$$

   where C is the straight line from $(0,1,2)$ to $(1,2,3)$.

4. (a) Define the Jacobian

$$\frac{\partial(u,v)}{\partial(x,y)},$$

where $u(x,y)$ and $v(x,y)$ are smooth functions.

(b) By changing the order of integration, or otherwise, find

$$\int_0^\infty \int_{\sqrt{y}}^\infty \exp(-\mu x^3) dx\, dy,$$

where μ is a positive constant. Show, by means of a sketch, the region of integration.

(c) By a suitable change of variables, or otherwise, find

$$\int_0^\infty \int_0^\infty \exp(-(x^2+y^2)) dx\, dy.$$

Show, by means of a sketch, the region of integration.

5. (a) State the divergence theorem carefully.

(b) Verify the divergence theorem for the vector field

$$\mathbf{A} = (x+y)\mathbf{i} + (x^2+xy)\mathbf{j} + z^2\mathbf{k},$$

and a unit radius ball centred at $(1,1,1)$. *Hint: You might find it helpful to move the ball so its centre is at the origin, as then it will be easy to use symmetry arguments for the integrals.*

6. (a) State carefully Green's theorem in the plane.

(b) Show that Green's theorem in the plane is a special case of Stokes' theorem, defining all your symbols carefully.

(c) Verify Green's theorem in the plane for

$$\oint_C (y dx + x(2+y) dy),$$

where C is the unit circle. *Hint: Use symmetry arguments where possible.*