UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M14B: Mathematical Methods 2

COURSE CODE	:	**MATHM14B**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is ***not*** *permitted in this examination.*

1. (a) State, without proof, the general formula for a Fourier series on $(-L, L)$, with $L > 0$, for a function $f(x)$, giving the expressions for the coefficients.
   (b) On $(-\pi, \pi)$, find the Fourier series of $f(x) = \exp(x)$.
   (c) Hence or otherwise, find the Fourier series of $g(x) = \sinh(x) + 2\cosh(x)$, on the range $(-\pi, \pi)$.

2. (a) Using subscript notation, what is the expression for

$$\varepsilon_{ijk}\varepsilon_{klm}$$

   in terms of δ_{il}, δ_{jm}, etc.?
   (b) Using subscript notation or otherwise, prove

$$\operatorname{div}(\phi \mathbf{A}) = (\operatorname{grad}\phi)\cdot\mathbf{A} + \phi \operatorname{div}\mathbf{A},$$

$$\operatorname{curl}\operatorname{curl}(\mathbf{A}) = \operatorname{grad}(\operatorname{div}\mathbf{A}) - \nabla^2\mathbf{A}.$$

   (c) Verify the results of the previous part (2b) of this question in the case

$$\phi = xy, \quad \mathbf{A} = (x^2, z^2, y^2).$$

3. Given that

$$\mathbf{F} = (x^2 + y + z)\mathbf{i} + (x + y)\mathbf{j} + xz^2\mathbf{k} + \operatorname{grad}((x + y)\exp(xz))$$

   evaluate the line integral

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

   from $(0,0,0)$ to $(1,2,3)$ along the following paths:
   (a) $C = C_1$ defined by $(x, y, z) = (t, 2t^2, 3t)$, for $0 \le t \le 1$,
   (b) $C = C_2$ which is the straight line from $(0,0,0)$ to $(1,2,3)$.

   Hint: You might find it helpful to consider $\mathbf{F}$ *in two parts, and use a different method for each part.*

4. (a) Define the Jacobian

$$\frac{\partial(x, y, z)}{\partial(u, v, w)},$$

where $x(u, v, w)$, $y(u, v, w)$ and $z(u, v, w)$ are smooth functions.

(b) Illustrate the spherical polar coordinate system by means of a sketch, and give the expressions for the Cartesian coordinates (x, y, z) in terms of the spherical polar coordinates.

(c) Determine the Jacobian, from your definition, for the change of coordinates from Cartesian coordinates to spherical polar coordinates.

(d) Find, by a using a suitable change of coordinates or otherwise,

$$\int_V f(x, y, z)\, dV,$$

where V is the unit ball and $f = [r \sin(\theta) \cos(\phi)]^4$ in terms of spherical polar coordinates.

Recall that for spherical polar coordinates, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

5. (a) State the divergence theorem carefully.

(b) Verify the divergence theorem for the vector field

$$\mathbf{A} = (1 + x + y)\mathbf{i} + (2 + xyz)\mathbf{j} + (x + y + z)^2\mathbf{k}$$

and the unit cube (i.e. $x \in [0, 1]$, $y \in [0, 1]$ and $z \in [0, 1]$).

6. (a) State Stokes' theorem carefully.

(b) Show that Stokes' theorem implies

$$\int_S \mathbf{n} \times \operatorname{grad} \phi \, dS = \int_C \phi \, d\mathbf{r}$$

where ϕ is a smooth scalar function, and S is an open (two-sided) surface with unit normal $\mathbf{n}$ and boundary C.

Hint: Consider curl $(\phi\mathbf{c})$, *where* $\mathbf{c}$ *is a constant vector.*

(c) Verify the result of the previous part (6b) of this question for the surface S defined by $z = x^2 + y^2$ and $z \leq 4$, and the scalar field $\phi = x$.